Kinematics in One Dimension

Chapter Two

Introduction

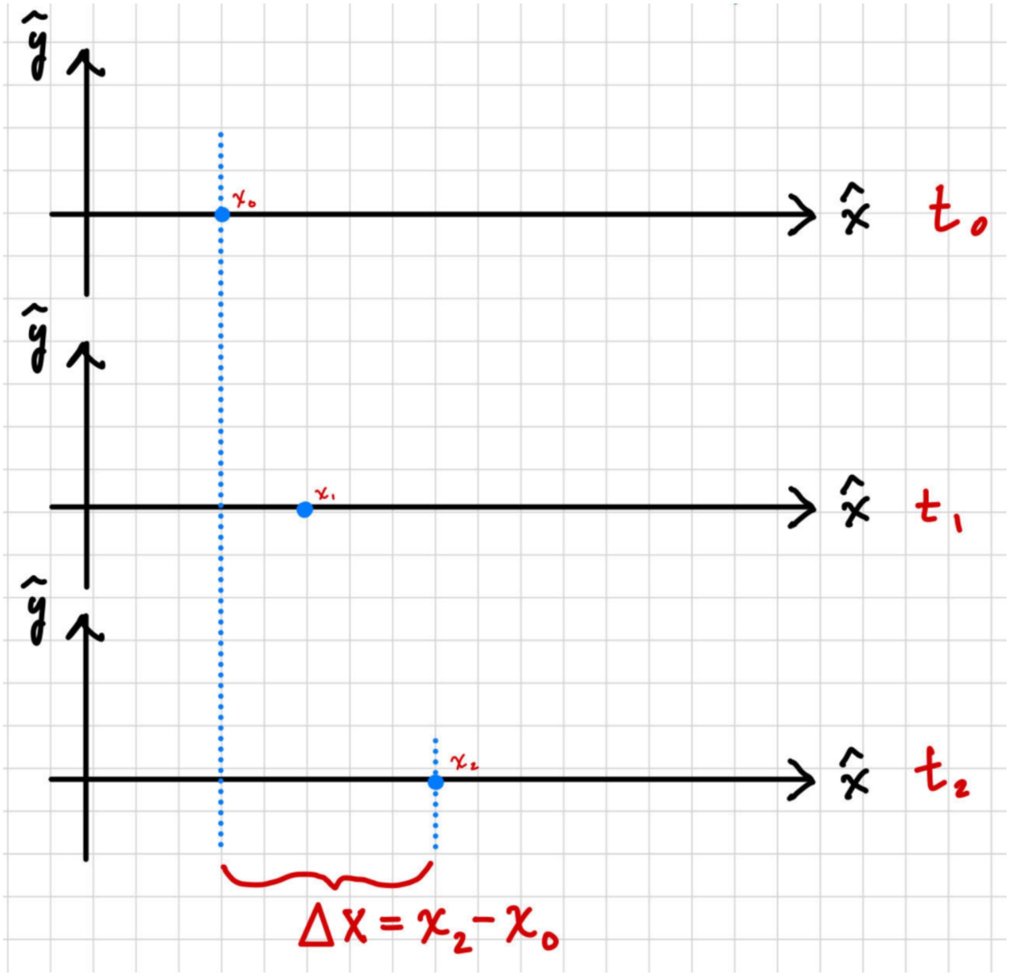
- Displacement
- Average speed vs Average Velocity
- Displacement and Average Velocity as Vectors
- Instantaneous Velocity
- Acceleration
- Kinematic Equations in 1D
- Free Fall
- Symmetry

Displacement in 1D

- To describe the motion of an object, we must be able to always specify the location of the object.
- Consider tracking a person who starts walking from some point.
    - Conceptually we know that they start at some point and then walk to some other point.
    - Since mathematics is the language of physics, we need to capture these points with coordinates.
- We can declare a starting position, and label it as x_0. This is when we start our timer. We would likely start it at 0, however, to keep things general let's label that as t_0.
- We mark down where the person is at intervals of time. At t_1 the person is at some location x_1 and so on.
- Once we are done, we can find the total displacement, Δx, by simply finding the difference between our last measurement of their position, x_3, and where we first measured them, x_0. $\Delta x = x_2 - x_0$. We could write this more generally as,

$$\Delta x = x - x_0$$

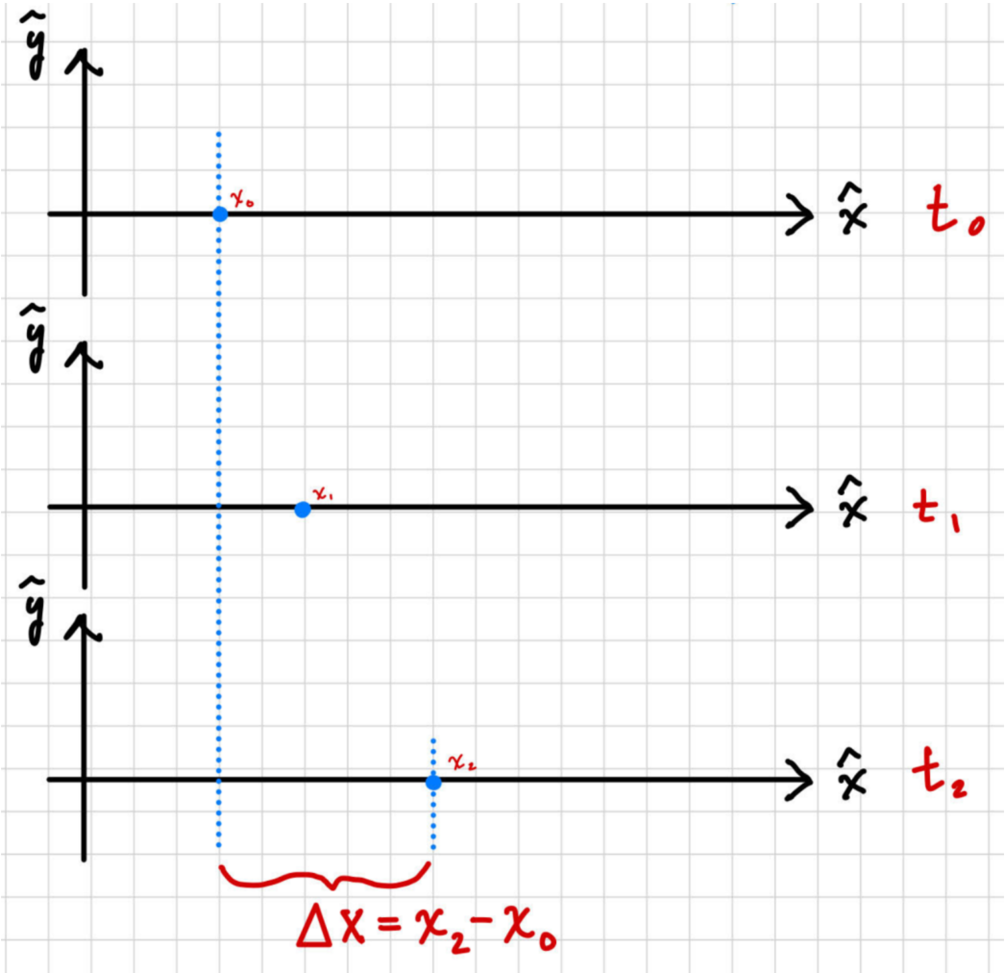
x is the final position, x_0 is the initial, or starting, position.

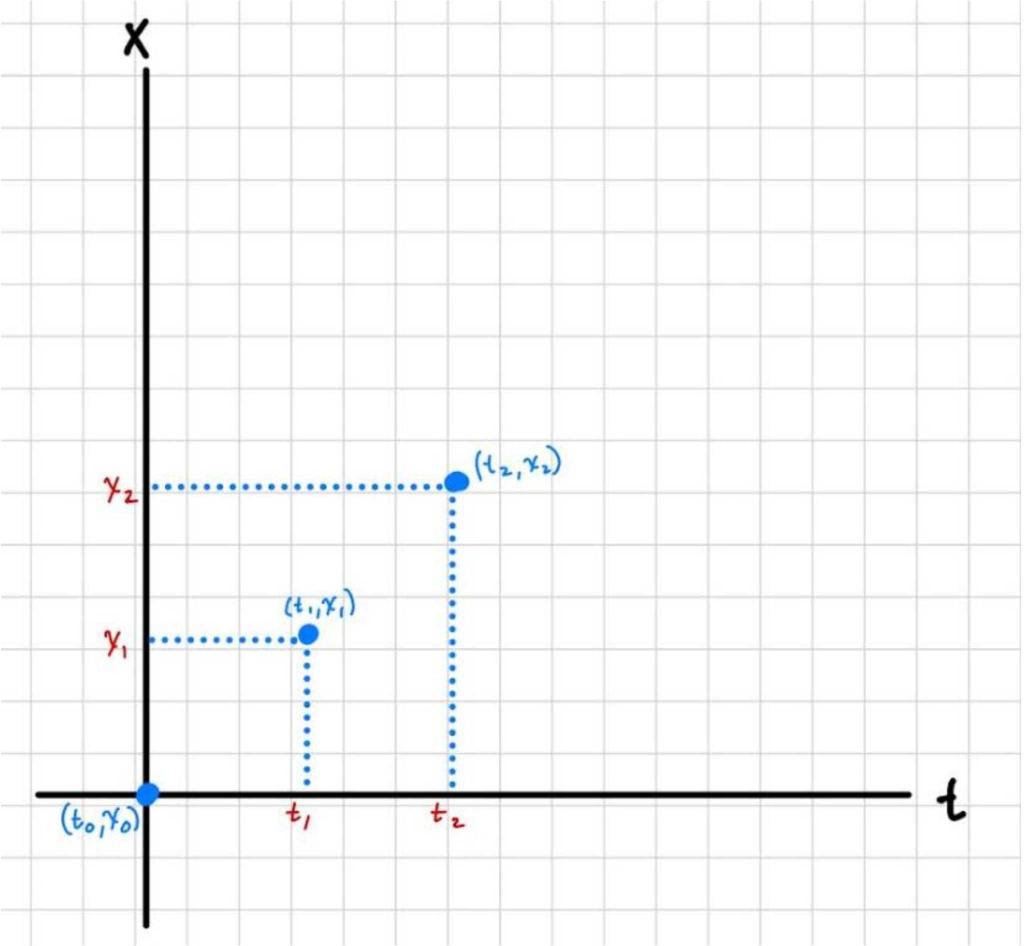


Team Activity 2.1

- In the previous example, our walker walked in a straight line. But what if they didn't?
- Gus starts at a position that we will say is 0m. He walks one meter forward, 2 meters backwards, and one meter forward again. What is the total distance that Gus walks? What is Gus's average displacement, Δx?

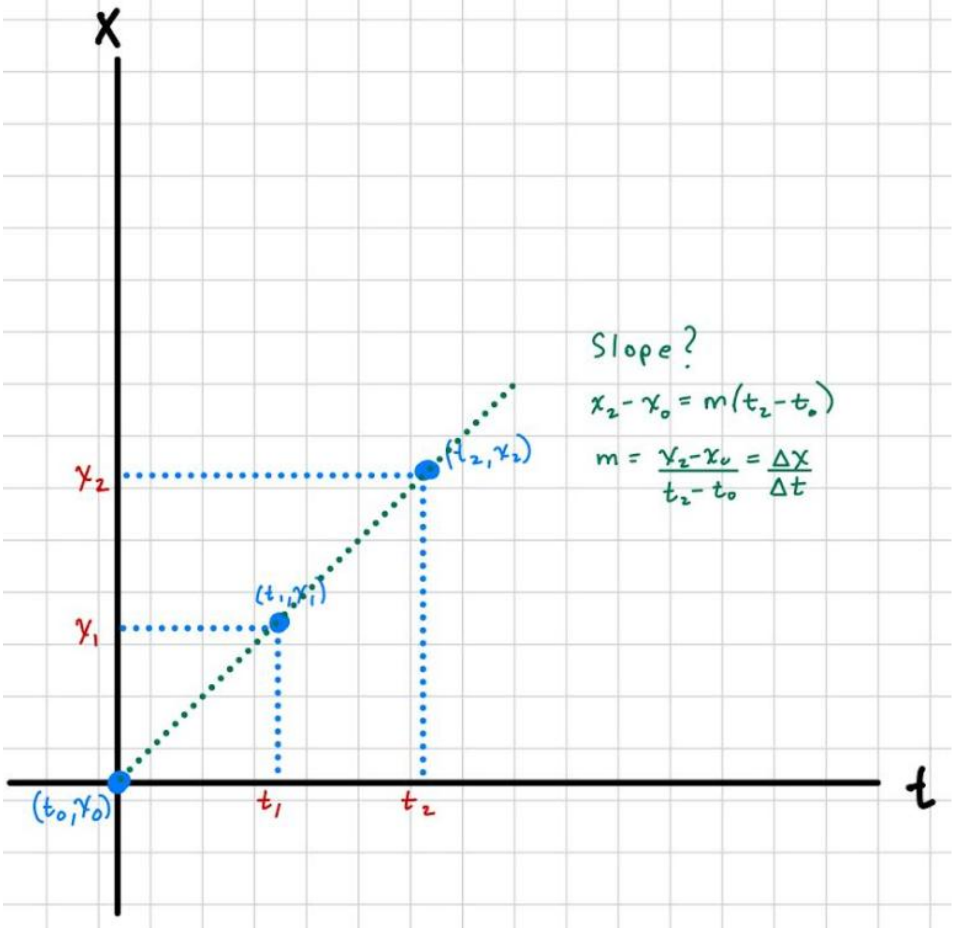
Speed and Velocity in 1D





Velocity in 1D

- We can connect the points with a line.
- From algebra, we know that the **slope** of a line is found by taking the difference in vertical axis values and dividing by the difference in horizontal axis values.
  - In algebra this was y vs. x, but we are doing x vs. t.
- The slope is given by $m = \frac{x_2 - x_0}{t_2 - t_0} = \frac{\Delta x}{\Delta t} = v$
- In physics this is the velocity.
- Because $\Delta x > 0$, this gives a positive velocity in the $\hat{x}$ direction.
- The magnitude, $v = \frac{\Delta x}{\Delta t}$, is the speed.
- **In this case, speed and velocity have the same value**, because the walker moves in a straight line in one direction.
- **But this won't always be true** — if the walker changes direction, the velocity (which tracks net displacement) and the speed (which tracks total distance) will be different.



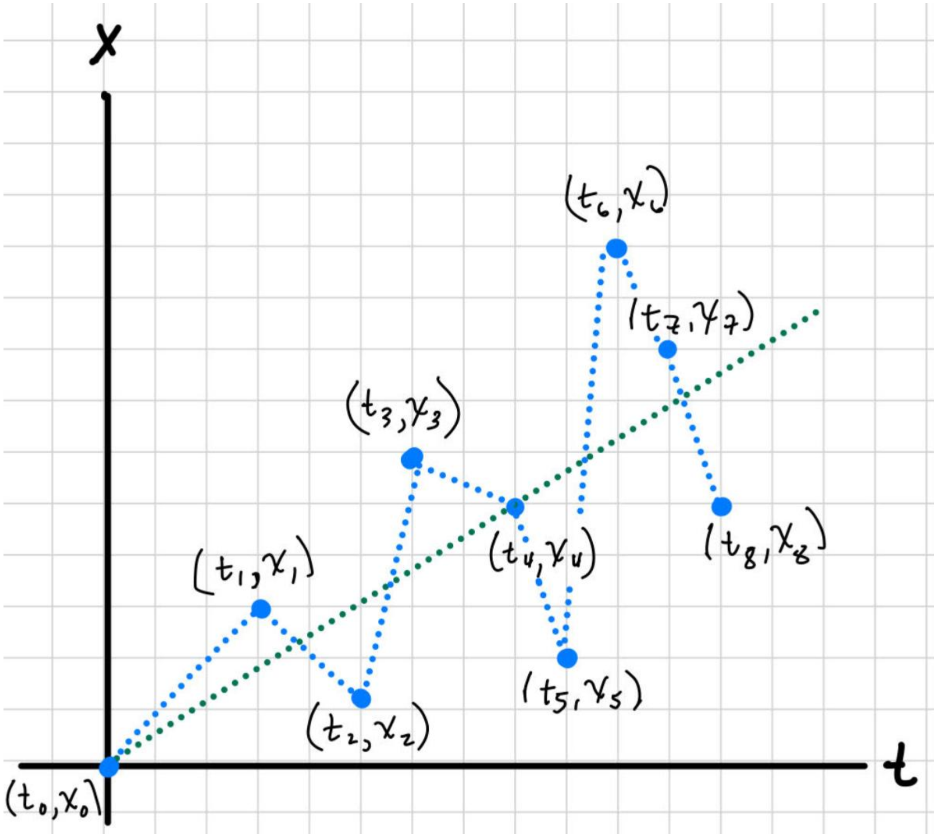
Next, let's look at what happens when the walker doesn't move in a straight line at constant speed...

Average Velocity vs Average Speed in 1D

- But what if we had to track Gus? Gus is a very confused walker.
- Notice that he walks back and forth and that his speed (the slope between points) changes.
- We can fit a best fit line to these points, and find the slope, and call that an average velocity, $\vec{v}_{avg} = \frac{x_{final} - x_{initial}}{t_{final} - t_{inital}} \hat{x}$.
- **Is the magnitude of this average velocity equal to Gus's average speed?** No.
- Gus did not walk in a straight line! He changed direction and backtracked.
- So, while his average velocity comes from this straight line, his average speed would be

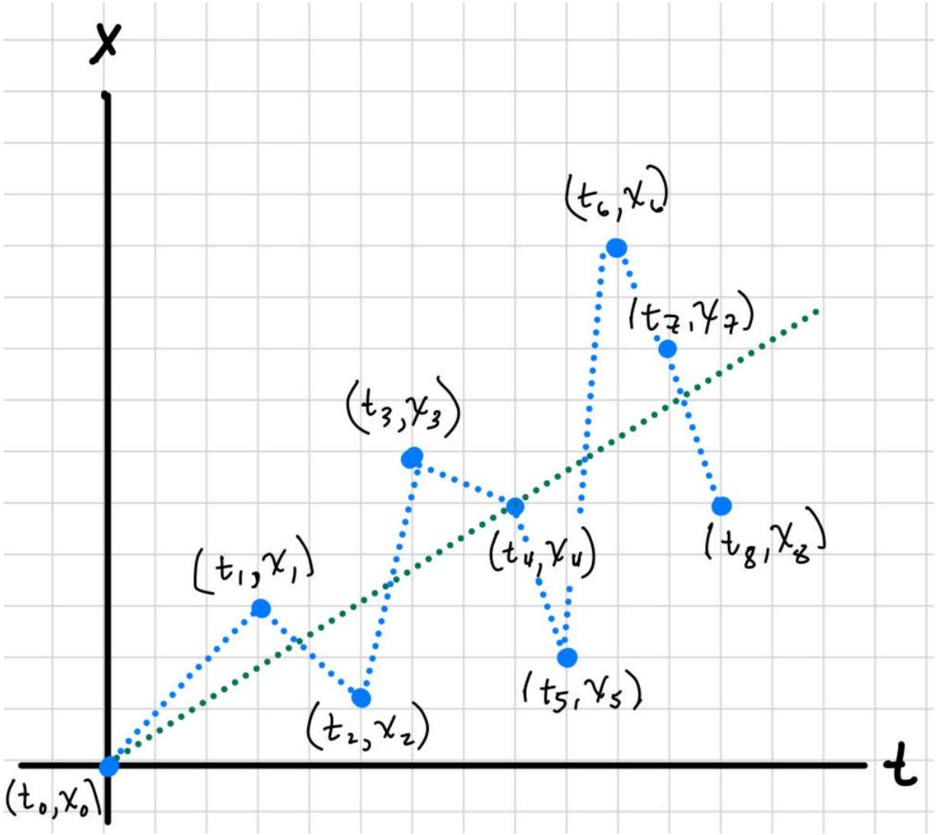
$$Average\ Speed = \frac{Total\ Distance\ Walked}{Total\ Time}$$

- **Average velocity** is based on **displacement** (net change in position).
- **Average speed** is based on **total distance traveled**, regardless of direction.



Example: Average Velocity vs Average Speed in 1D

- Each tick on the graph paper along x is 1 meter. Along t is 1 second.
  - We measure the distance from x = 0 at each interval of time.
  - (x, t) = (0,0), (3s, 3m), (5s, 1m), (6s, 6m), (8s, 5m), (9s, 2m), (10s, 10m), (11s, 8m), (12s, 5m)
- What is Gus's total average speed (based on the total distance he walked indicated by the dotted blue line)?
  - $\frac{(|3-0|+|1-3|+|6-1|+|5-6|+|2-5|+|10-2|+|8-10|+|5-8|)m}{12s} = \frac{27m}{12s} = 2.25\frac{m}{s}$
- What is the magnitude of his average velocity?
  - $\frac{\left((3-0)+(1-3)+(6-1)+(5-6)+(2-5)+(10-2)+(8-10)+(5-8)\right)m}{12s} = \frac{5m}{12s} = 0.42\frac{m}{s}$



Notice: *Even though Gus walked 27 meters total, his net displacement was only 5 meters — so his average speed is much greater than the magnitude of his average velocity.*

Team Activity: Concept Check 2.1

- Gus's average velocity was 0.42 m/s. If Gus had returned to his starting position at x = 0, what would his average velocity be?

Displacement as a Vector in 1D

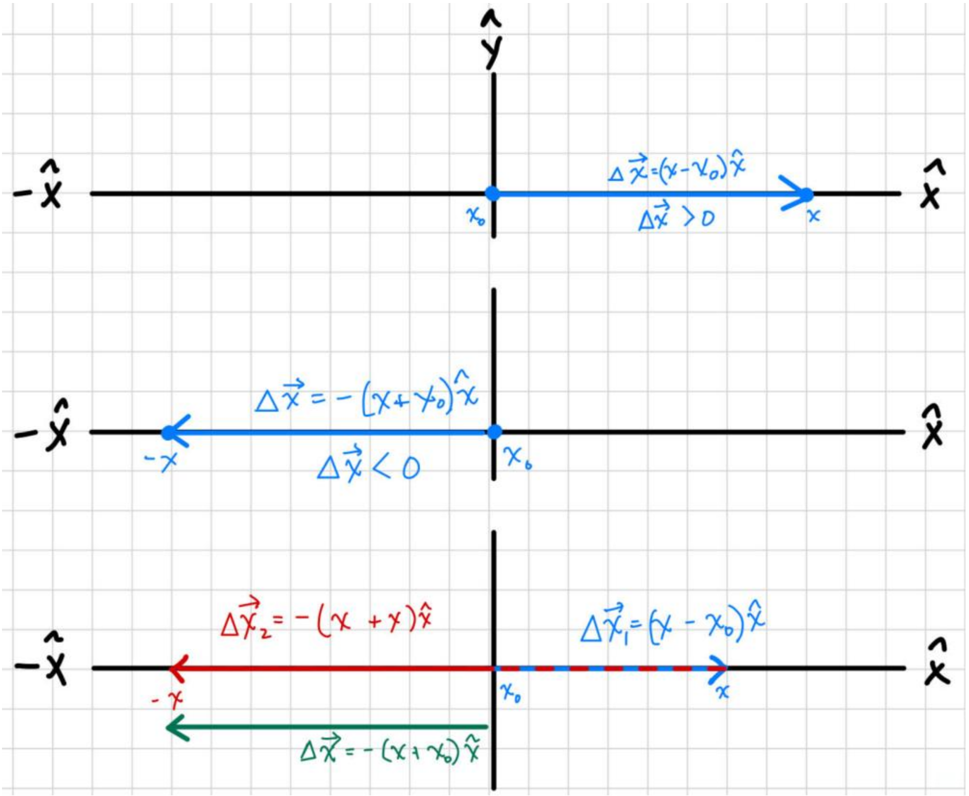
- Recall in our team activity, we discovered that a person who walks away from a starting position, only to return to it, has a displacement of zero. This is not how scalars behave. This is how vectors behave!
- Displacement isn't just "how far you moved" — it's **where you ended up relative to where you started**.
- Displacement has:
  - **Magnitude**: how far
  - **Direction**: where
- We can formalize our definition of displacement,

$$\Delta\vec{x} = \vec{x} - \vec{x}_0$$

- In one dimension, this is just,

$$\Delta\vec{x} = (x - x_0)\hat{x}$$

- If $\Delta\vec{x} > 0$ it represents a displacement vector pointing in the positive $\hat{x}$ direction.
- If $\Delta\vec{x} < 0$ it represents a displacement vector pointing in the negative $\hat{x}$ direction.

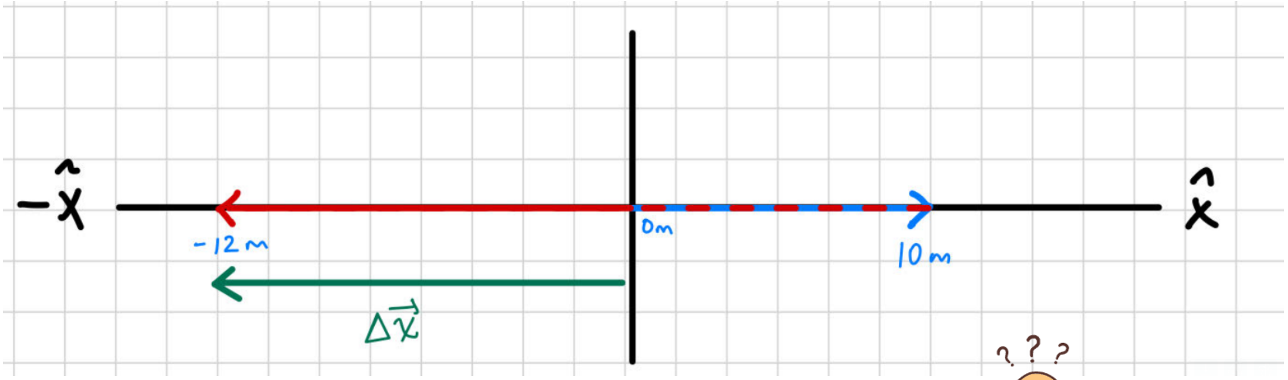


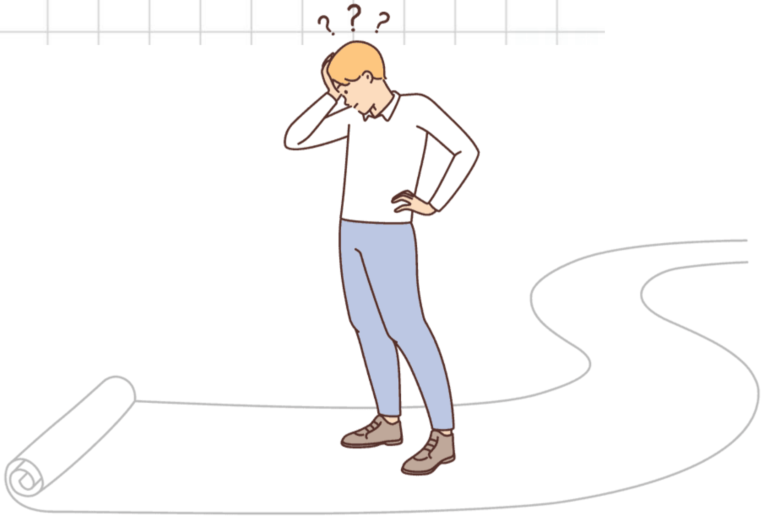
$$\Delta\vec{x} = \Delta\vec{x}_2 + \Delta\vec{x}_1 = -(x + x)\hat{x} + (x - x_0)\hat{x}$$

$$= -(2x - x + x_0)\hat{x} = -(x + x_0)\hat{x}$$

Team Activity 2.2

- Gus, that confused scoundrel, leaves his house to walk to work. When he gets to 10 meters, he realizes that he is heading in the wrong direction and proceeds to walk 22 meters in the opposite direction. What is Gus's total displacement vector, $\Delta\vec{x}$?



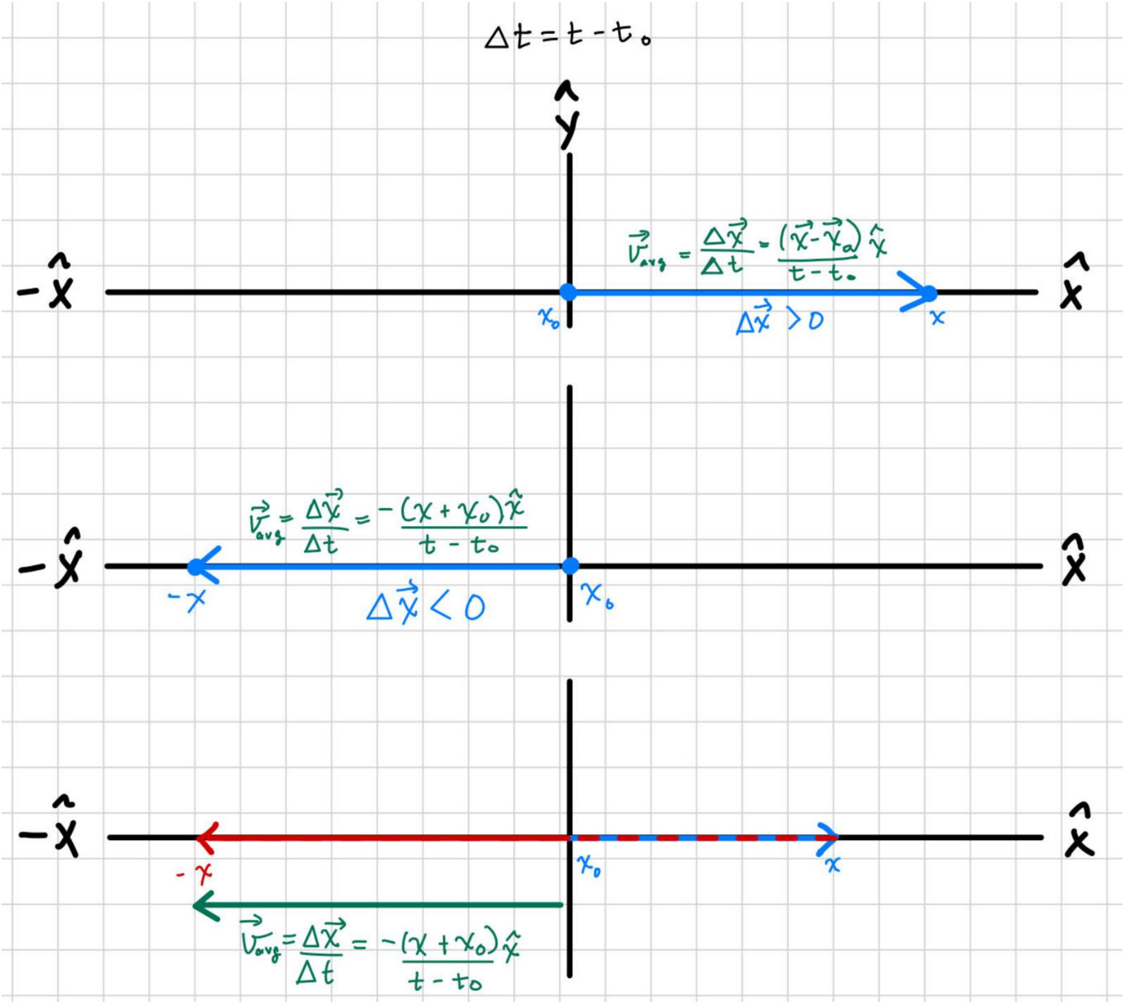


Average Velocity From Displacement 1/2

- In our earlier examples, we say that average velocity could be
  - Positive, negative, or even zero – depending on where Gus ended up.
- That's because average velocity, like displacement, is a vector
- Formal Definition:

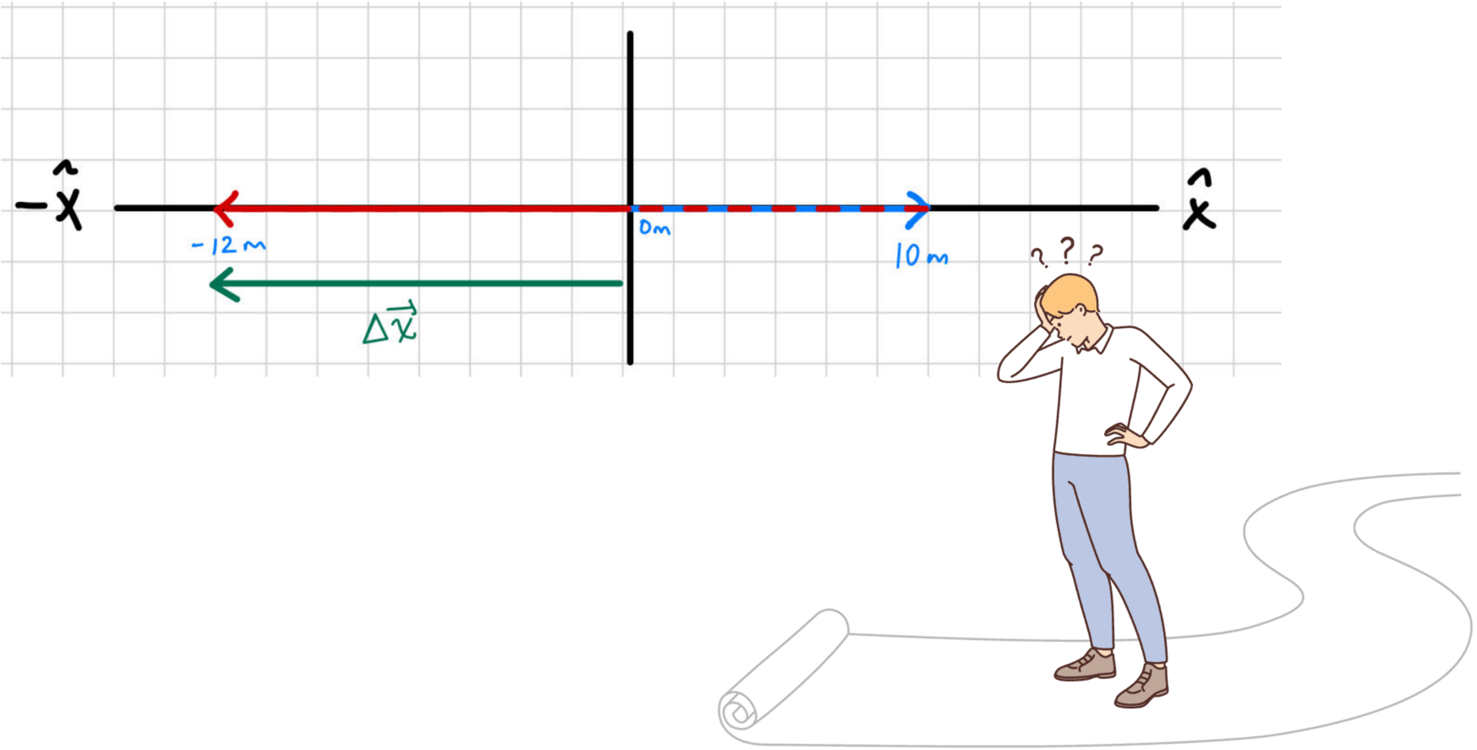
$$\vec{v}_{avg} = \frac{\Delta\vec{x}}{\Delta t}$$

- It describes how quickly and in what direction the position changes over time.
- The direction of $\vec{v}_{avg}$ is the same as the direction of $\Delta\vec{x}$.
- It depends only on the starting and ending positions and times – not the path.
- As the plot shows, we can use the displacement vector, $\Delta\vec{x}$, to obtain the average velocity vector, $\vec{v}_{avg}$ given the the elapsed time during the motion, $\Delta t = t - t_0$.

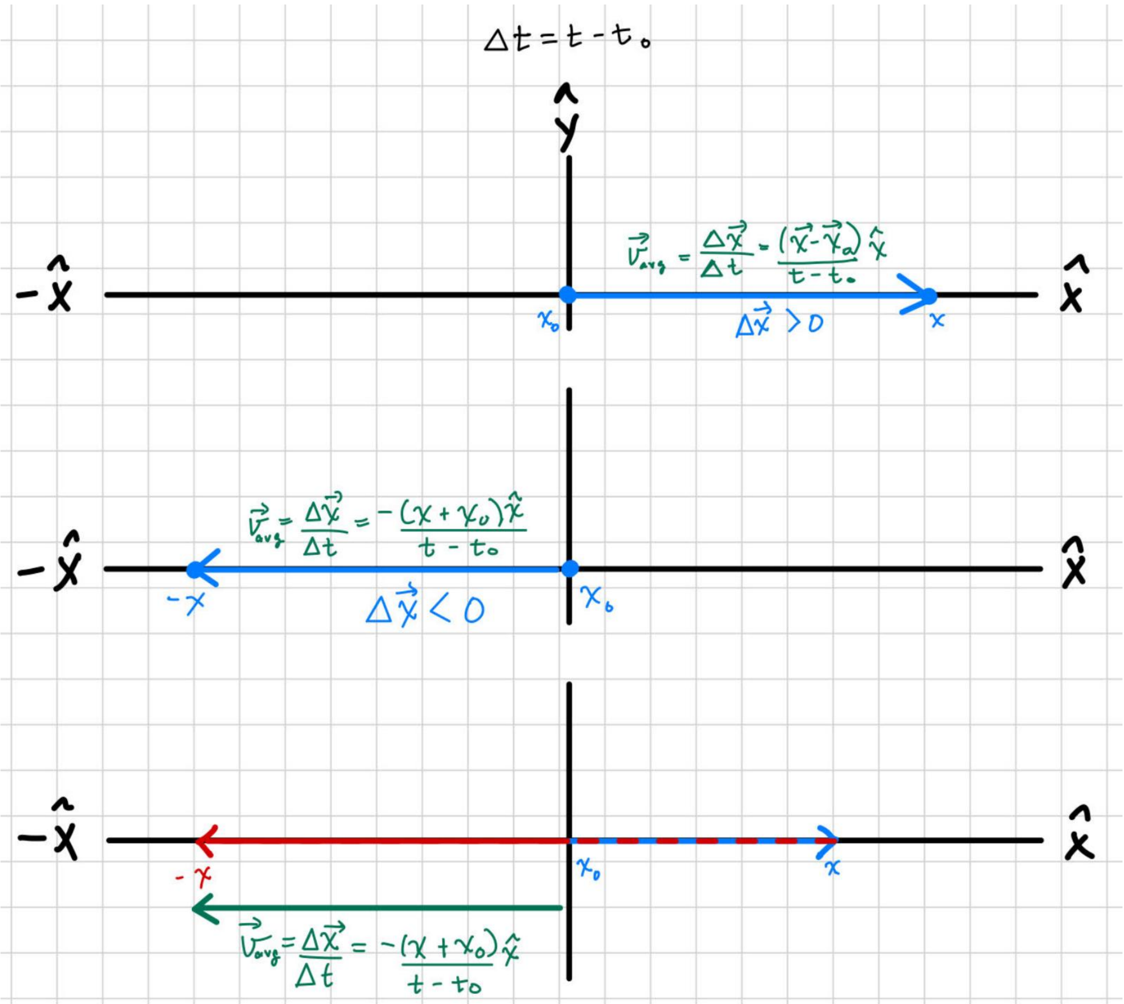


Team Activity 2.3

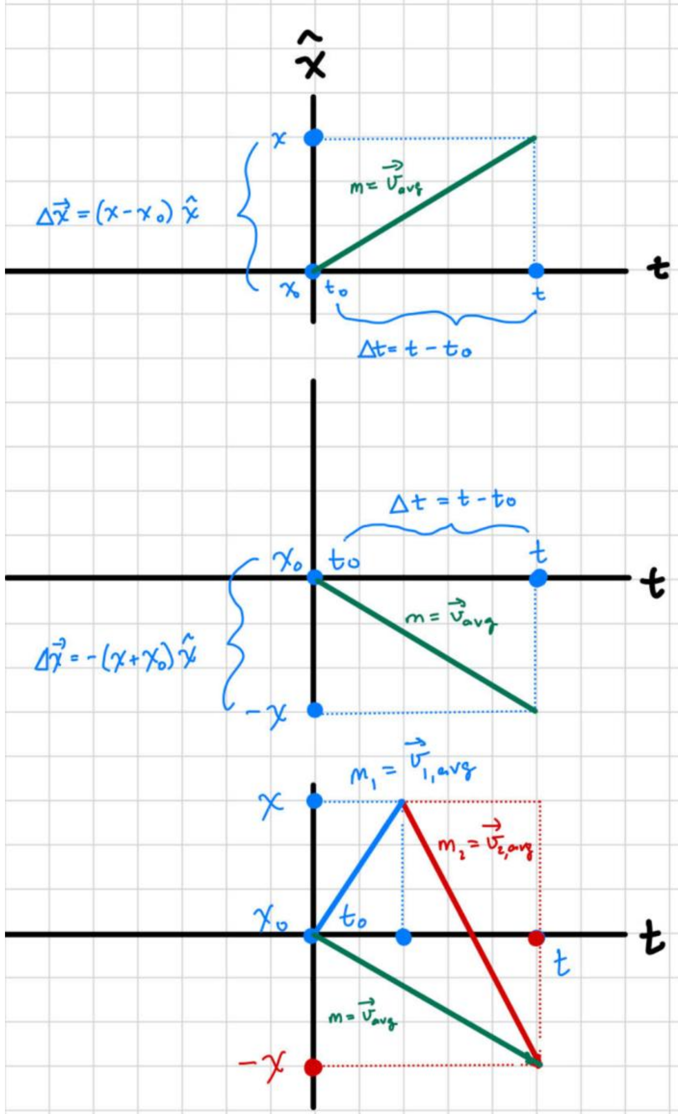
- Gus, that confused scoundrel, leaves his house to walk to work. When he gets to 10 meters, he stops to think for 5 minutes and realizes that he is heading in the wrong direction and proceeds to walk 22 meters in the opposite direction. The total trip took 6 minutes. What is Gus's average velocity, $\vec{v}_{avg}$?



Average Velocity from Displacement 2/2



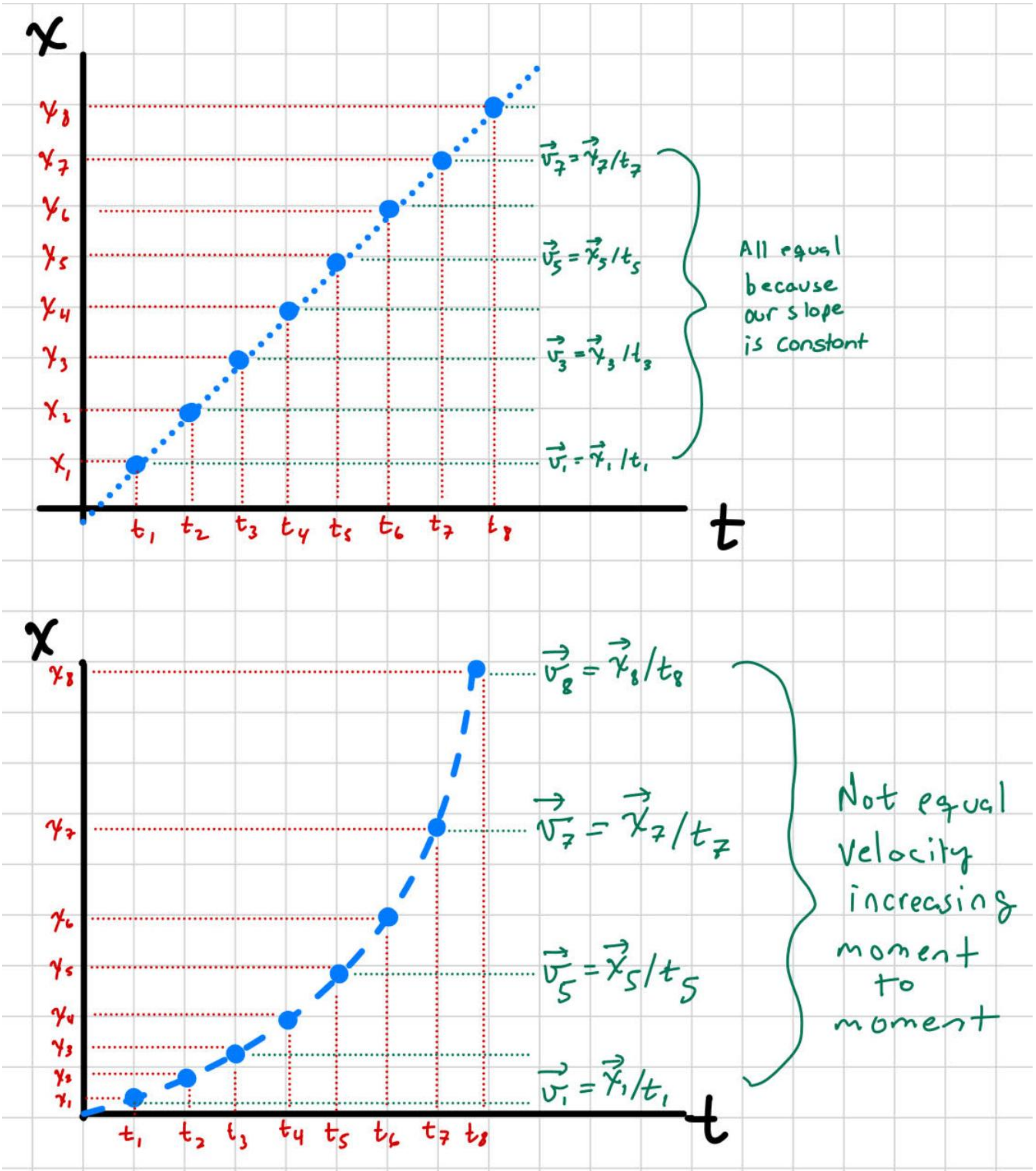
Displacement as a vector



x vs. t gives information about our v vectors

Instantaneous Velocity 1/2

- Up until now we have considered average velocities. We don't really know the moment-to-moment velocity between the points x and x_0 so we computed an average.
- But let us now consider just that. A plot of x vs. t where each point along it represents the exact velocity at a specific moment in time.
- In both plots, each blue dot represents the instantaneous velocity at a moment in time — what the object is doing right then, not averaged over a time span.
- Plot 1: Constant Velocity. If the object's motion is steady — no speeding up or slowing down — then the velocity remains constant. The x vs. t graph is a straight line, and each dot has the same slope (velocity).
- Plot 2: Changing Velocity. Each dot gives a steeper slope than the one before — showing that the object is speeding up. This kind of motion will soon lead us to a new concept: acceleration



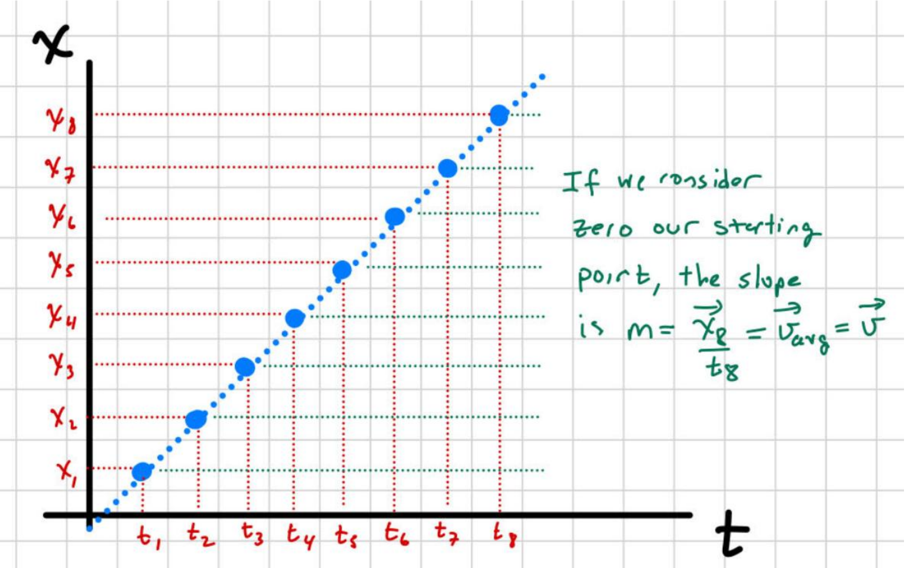
Instantaneous Velocity 2/2

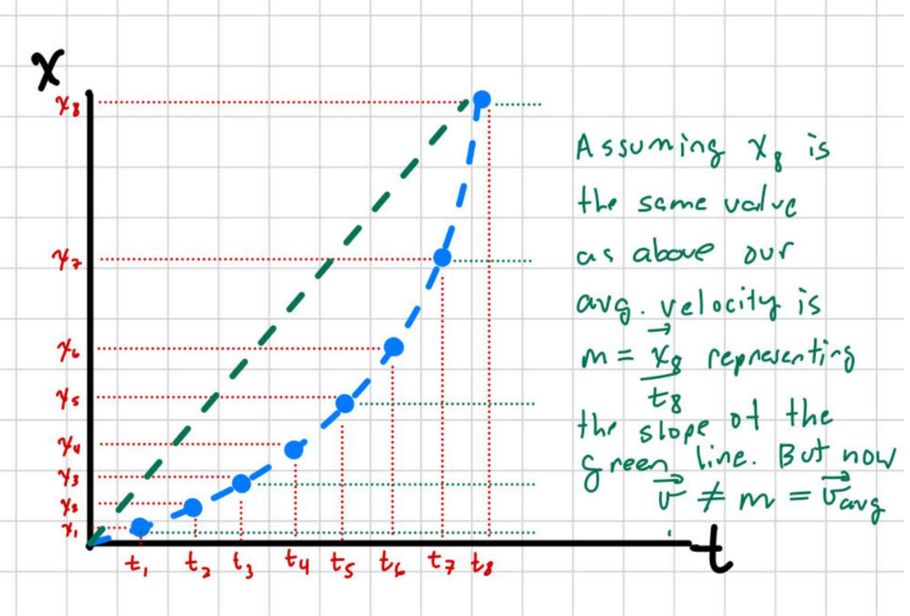
- First Plot: If we consider zero to be our starting point, the slope is

$$m = \frac{\vec{x}_8 - 0}{t_8 - 0} = \frac{\vec{x}_8}{t_8} = \vec{v}_{avg} = \vec{v}$$

That is, the average velocity is equal to the instantaneous velocity!

- Second plot: We can draw a straight line from (0,0) to (t_8, x_8) to represent the average velocity – but this line doesn't match the actual motion. The blue curve shows a changing velocity, and now the average and instantaneous velocities are no longer equal.



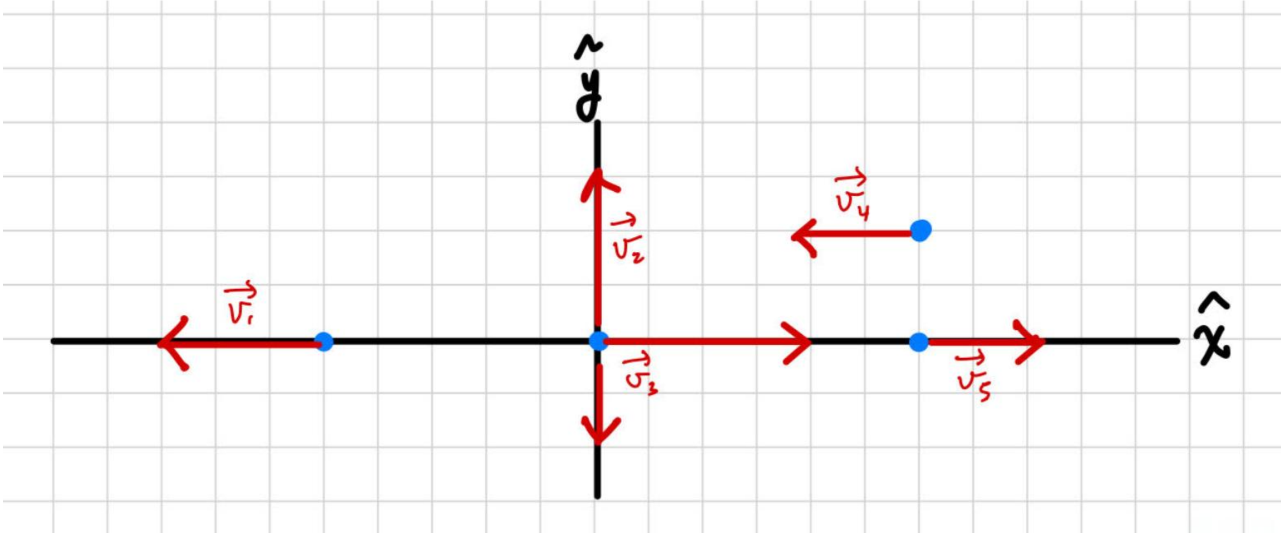


Team Activity: Concept Check 2.2

- Watch out – someone gave Gus a driver's license! Gus is merging onto a highway and therefore increasing his speed every second as he does so. If we compute his average velocity, it will be equal to his instantaneous velocity at each second – true or false?

Velocity as a 1D Vector

- You have probably noticed that we have avoided drawing velocity as a vector. But why?
  - We draw vectors in spatial coordinate systems, such as cartesian (y vs. x).
  - In that coordinate system, a vector extending from a point represents the instantaneous velocity, direction and magnitude, at that point.
  - However, with average velocity, although a vector, we cannot encode that it is an average over instantaneous velocities at all points.
  - If the speed is constant, then as we have seen the instantaneous velocity is the average velocity and we could draw a vector in that specific scenario.
- Though in 1D it isn't very useful because it is entirely along a single direction (in the plot, either completely in x or completely in y).

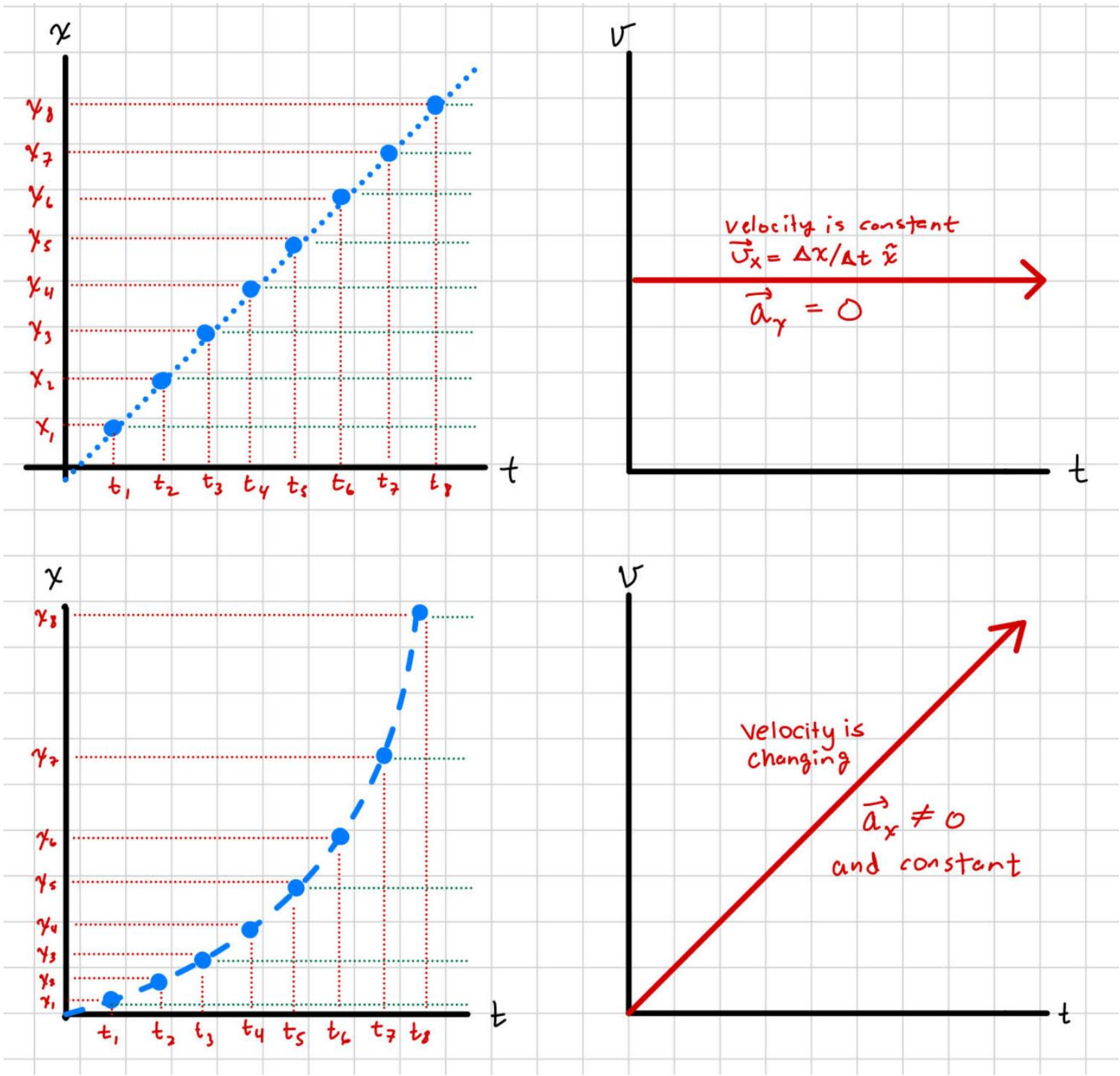


- In the above plot, we show 5 1D vectors. Each one represents the velocity at the blue point from which it extends.
- The direction it extends is the direction of the velocity vector.
- But the length of the vector, while it does correspond to the magnitude of each velocity, is measured on its own scale, and not by the tick marks on the x or y axis.
- Furthermore, while the vector does appear to pass through neighboring points, it does not mean that the velocity is the same at those points.

Acceleration in 1D

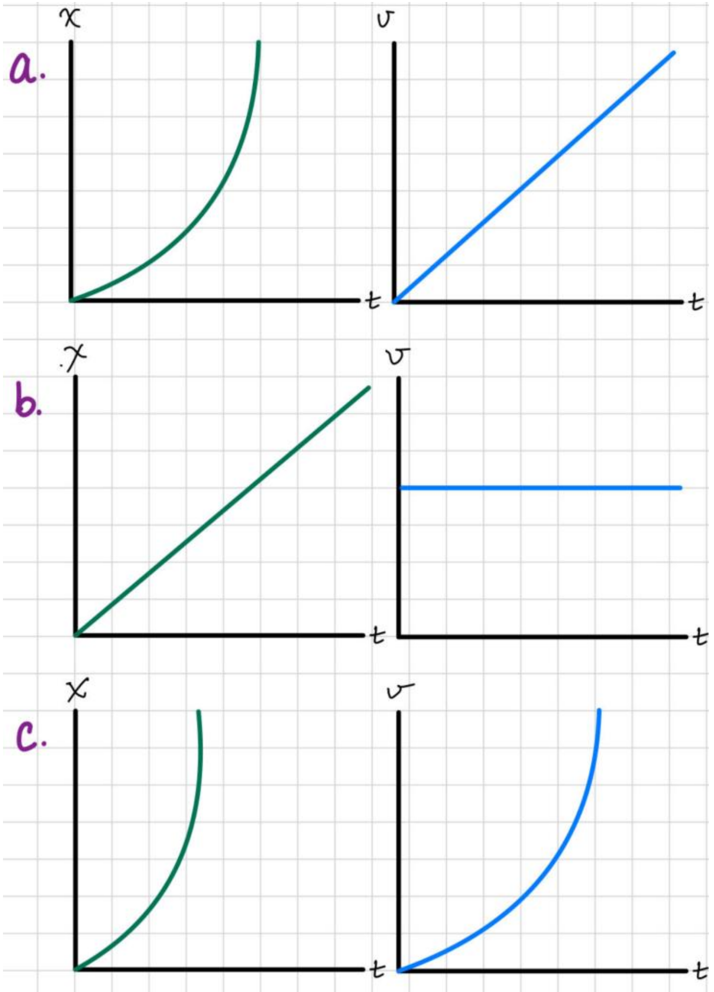
- Top row: $\vec{a}_x = 0$.
  - $\vec{v}_x$ is constant.
  - $\Delta\vec{x}$ is linear.
- The bottom row: $\vec{a}_x \neq 0$, and constant.
  - $\vec{v}_x$ is not a constant and is linear.
  - $\Delta\vec{x}$ is non-linear.
- Acceleration is slope of the v vs. t plot!
- Formal definition of acceleration in 1D,

$$\vec{a}_x = \frac{\vec{v}_{x,f} - \vec{v}_{x,0}}{t_f - t_0} = \frac{\Delta\vec{v}_x}{\Delta t}$$



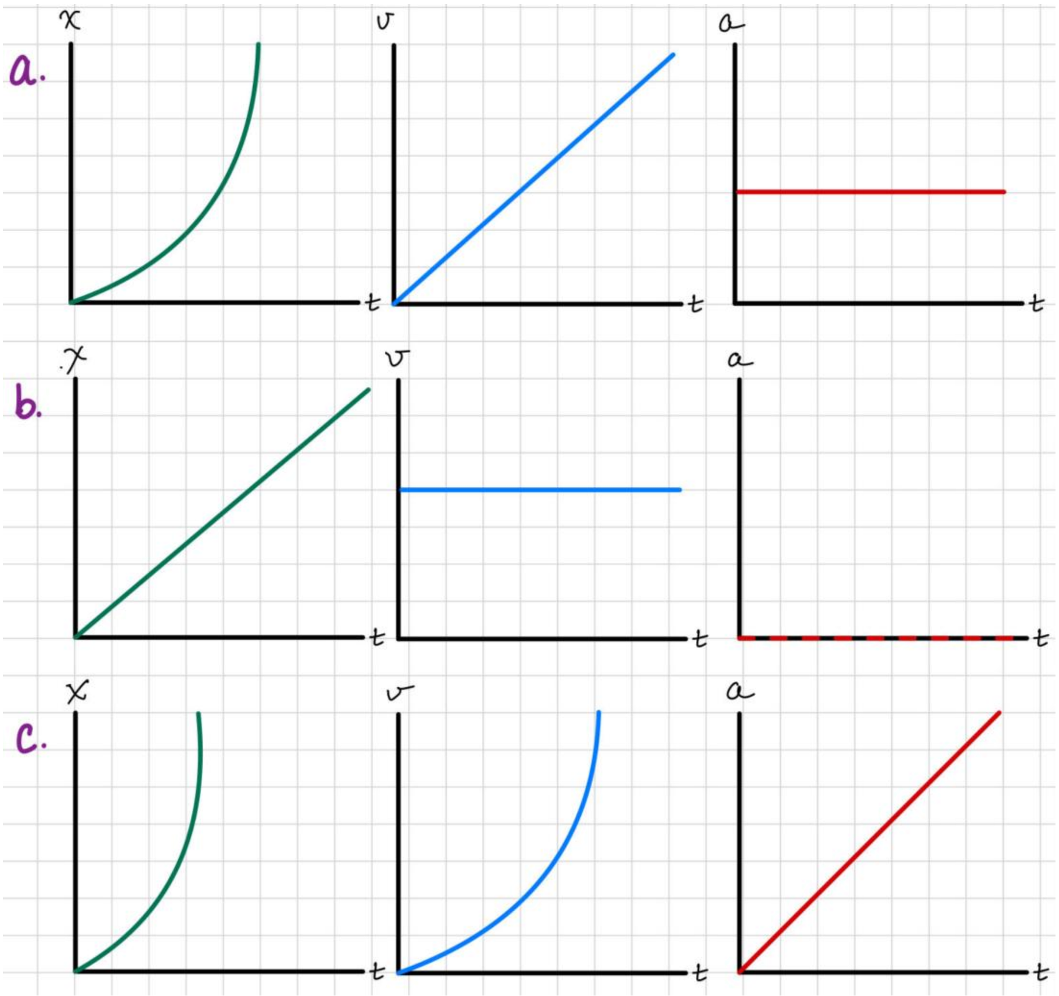
Team Activity: Concept Check 2.3

- Consider three different scenarios labeled as a, b, and c. Label scenarios a, b, and c with the following
  - $\vec{a}_x = 0$.
  - $\vec{a}_x \neq 0$, not constant.
  - $\vec{a}_x \neq 0$, constant.

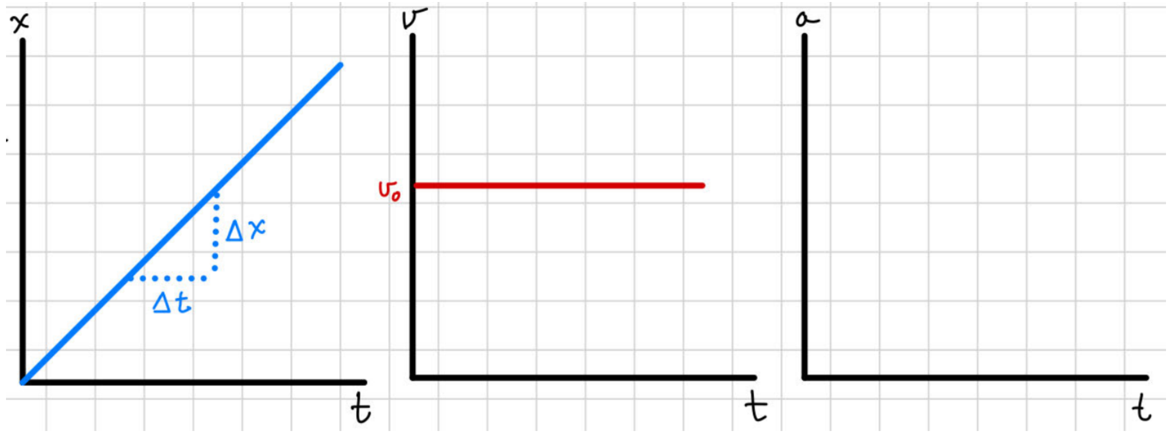


Team Activity: Concept Check 2.3 cont.

- Consider three different scenarios labeled as a, b, and c. Label scenarios a, b, and c with the following
  - $\vec{a}_x = 0$.
  - $\vec{a}_x \neq 0$, not constant.
  - $\vec{a}_x \neq 0$, constant.



Equations of Kinematics for Constant Velocity



Consider the case when $\vec{v}_x$ is constant ($\vec{a}_x = 0$).

$$\vec{v} = \frac{\Delta\vec{x}}{\Delta t} = \frac{(x_f - x_0)\hat{x}}{t_f - t_0}$$

For simplicity, let $t_0 = 0$, $t_f = t$, $x_f = x$. Also, we will write in "component form".

$$v_x = v_0 = \frac{x - x_0}{t} \quad \rightarrow \quad x - x_0 = v_0 t \quad \text{(p1)}$$

Note that Eq. (p1) only holds if acceleration is zero. But what if its not?

1D Equations of Kinematics for Constant Acceleration

Consider the case when $\vec{a}_x$ is constant ($\vec{a}_x \neq 0$).

Eq. p1, $x - x_0 = v_0 t$, is no longer valid because velocity is no longer constant. But we can fix it.

Compute an average velocity $\bar{v}_x = \frac{v_x - v_0}{2}$ and substitute into Eq.p1 for v_0.

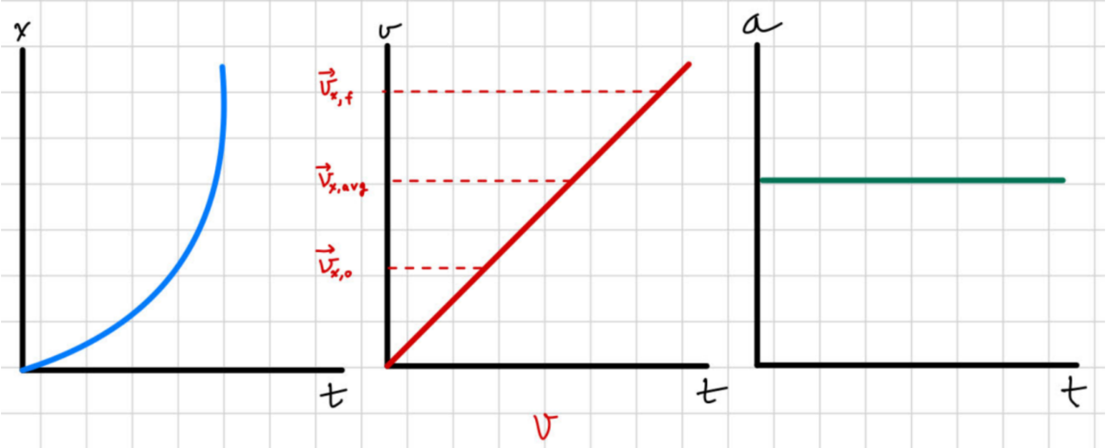
$$x - x_0 = \left(\frac{v_x - v_0}{2}\right) t = \frac{1}{2} v_x t - \frac{1}{2} v_0 t \qquad \text{(p2)}$$

Consider acceleration in component form

$$a_x = \frac{\Delta v_x}{t} = \frac{v_x - v_0}{t} \quad \rightarrow \quad \boxed{v_x = v_0 + a_x t} \qquad (1)$$

Substitute Eq. 1 into Eq. p2

$$x - x_0 = \frac{1}{2}(v_0 + a_x t)t - \frac{1}{2} v_0 t \quad \rightarrow \quad \boxed{x - x_0 = v_0 t + \frac{1}{2} a_x t^2} \qquad (2)$$



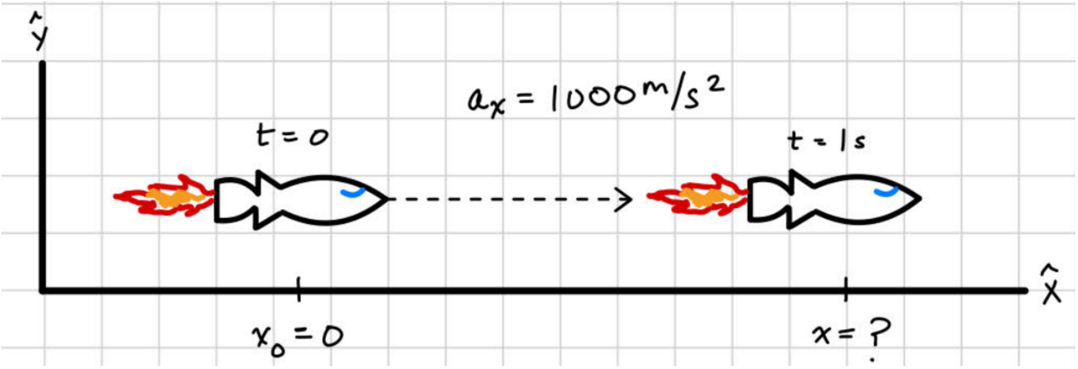
We can combine Eq. 1 and Eq. 2 to get a third useful equation,

$$x - x_0 = v_0 t + \frac{1}{2}(v_x - v_0)t = \frac{1}{2}(v_x + v_0)t$$

$$\boxed{x - x_0 = \frac{1}{2}(v_x + v_0)t} \qquad (3)$$

Example Problem 1D Kinematics I

Consider a spaceship moving in a straight line through space. At $t = 0$ s it begins at the starting point $x_0 = 0$ from rest. It is accelerating at $1000\ m/s^2$. What are the total distance traveled and the final velocity?



Step one: Draw a diagram

Step two: List out your known variables

$$t_0 = 0, \quad x_0 = 0, \quad v_0 = 0, \quad t = 1s, \quad a_x = 1000\ m/s^2$$

Step three: List out your unknown variables

$$x, \quad v_x$$

Step four: Write out your kinematic equations and simplify and choose the relevant equations.

Step four: Write out your kinematic equations and simplify and choose the relevant equations.

$$v_x = \cancel{v_0} + a_x t \quad \rightarrow \quad v_x = a_x t$$

$$x - \cancel{x_0} = \cancel{v_0 t} + \frac{1}{2}a_x t^2 \quad \rightarrow \quad x = \frac{1}{2}a_x t^2$$

$$x - \cancel{x_0} = \frac{1}{2}(v_x + \cancel{v_0})t \quad \rightarrow \quad x = \frac{1}{2}v_x t$$

Step Five: Solve

$$v_x = (1000\ m/s^2)(1s) = \boxed{1000\ m/s}$$

$$x = \frac{1}{2}(1000\ m/s^2)(1s)^2 = \boxed{500\ m}$$

Example Problem 1D Kinematics II

A spaceship launches straight up from the Earth's surface. It starts from rest on the launch pad, 10m above the ground. After 1s the rocket is 510m from the ground. What is the final velocity and the acceleration excluding the acceleration due to gravity.

Step one: Draw a diagram

Step two: List out your known variables

$$t_0 = 0, \quad y_0 = 10m, \quad v_0 = 0, \quad t = 1s, \quad y = 510m, \quad g = -9.81\ m/s^2$$

Step three: List out your unknown variables

$$a_R, v_y$$

Note that there is a downward acceleration due to Earth's gravitational pull. We will discuss this in a later chapter but for now,

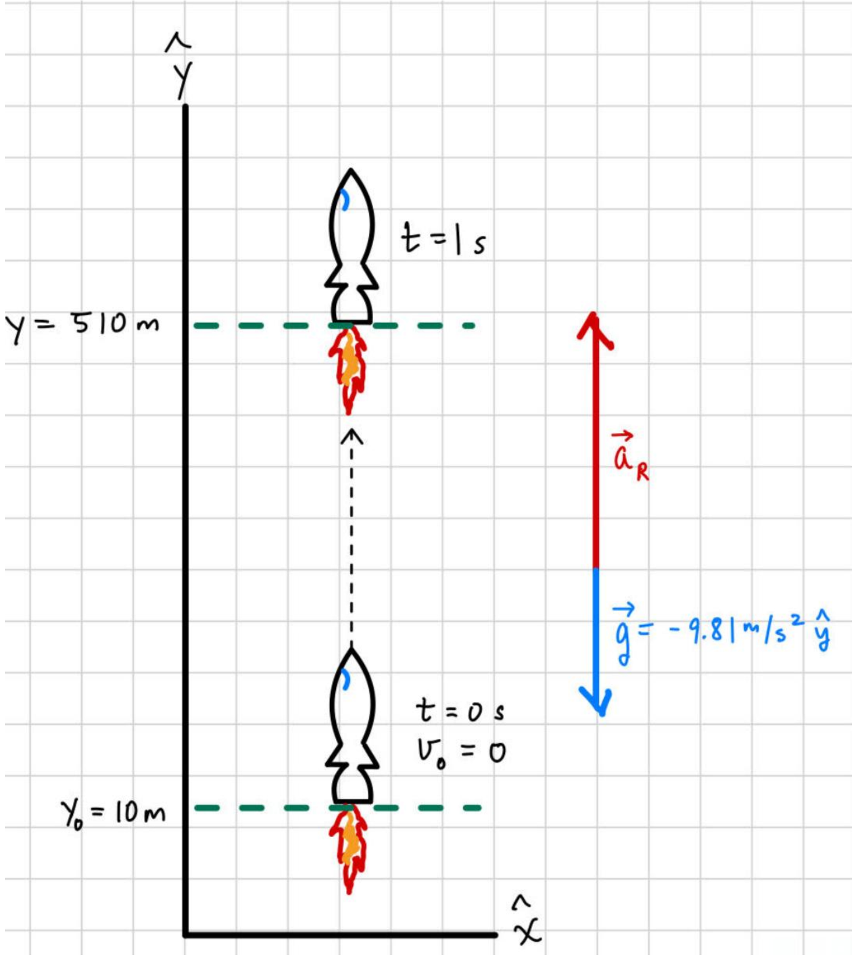
$$a_y = a_R + g$$

Step four: Write out your kinematic equations and simplify and choose the relevant equations.

$$v_y = \cancel{v_0} + a_y t \quad \rightarrow \quad v_y = (a_R + g)t \quad \rightarrow \quad a_R = \frac{v_y}{t} - g = \frac{1000m/s}{1s} + 9.81ms^2 = \boxed{1009.81m/s^2}$$

$$y - y_0 = \cancel{v_0}t + \frac{1}{2}a_y t^2 \quad \rightarrow \quad y - y_0 = \frac{1}{2}(a_R + g)t^2$$

$$y - y_0 = \frac{1}{2}(v_y + \cancel{v_0})t \quad \rightarrow \quad y - y_0 = \frac{1}{2}v_y t \quad \rightarrow \quad v_y = \frac{2(y - y_0)}{t} = \frac{2(510 - 500)m}{1s} = \boxed{1000\text{m/s}}$$



Notice that the rocket's acceleration is 9.81 m/s² more than the previous problem – indicating it overcoming Earth's gravity!

Free Fall

- **What happens when you drop a hammer and a feather?**
- On **Earth**, the hammer hits the ground quickly while the feather drifts down slowly.
- But on the **Moon**—where there's no atmosphere—they fall side by side and land at the same time.
- Why?
  - Because without air resistance, **both experience the same acceleration due to gravity an no other forces**.
- When gravity is the only force acting on an object, we call this motion **free fall**.

Team Activity: Concept Check 2.4

Gus took a wrong turn and ended up on the Moon. While driving, he gets distracted by a bouncing basketball and drives straight off a cliff. At that exact moment, from the same height, the basketball also bounces off the cliff.

Question: Which hits the ground first—Gus's car or the basketball?

Example: Free Fall

A rocket launches from the moon and makes it to a height of h when suddenly it suffers engine failure and begins to fall backward. After about 100 s it hits the launch pad located 10m from the surface of the moon. The acceleration due to the Moon's gravity is 1.625 m/s^2 downward. What is h? What was the final velocity just before it crashed landed?

Step one: Draw a diagram

Step two: List out your known variables

$$t_0 = 0, \quad y = 10m, \quad v_0 = 0, \quad t = 100s, \quad a_y = g_m = -1.625\ m/s^2$$

Step three: List out your unknown variables

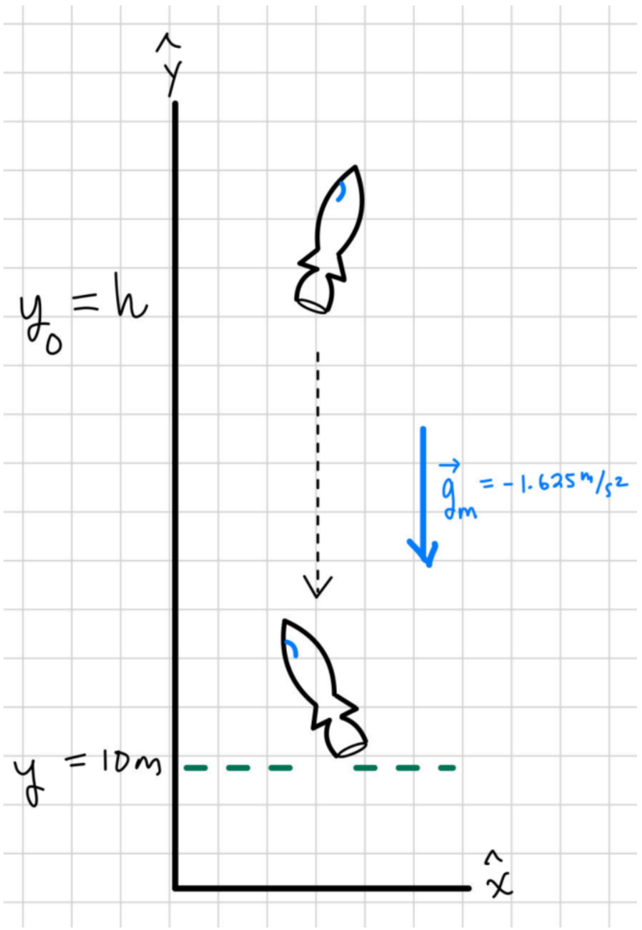
$$y_0 = h = ?, v_y = ?$$

Step four: Write out your kinematic equations and simplify and choose the relevant equations.

$$v_y = \cancel{v_0} + a_y t \quad \rightarrow \quad v_y = g_m t = \left(-1.625 \frac{m}{s^2}\right)(100s) = -162.5\ m/s$$

$$y - y_0 = \cancel{v_0} t + \frac{1}{2} a_y t^2 \quad \rightarrow \quad y - h = \frac{1}{2} g_m t^2$$

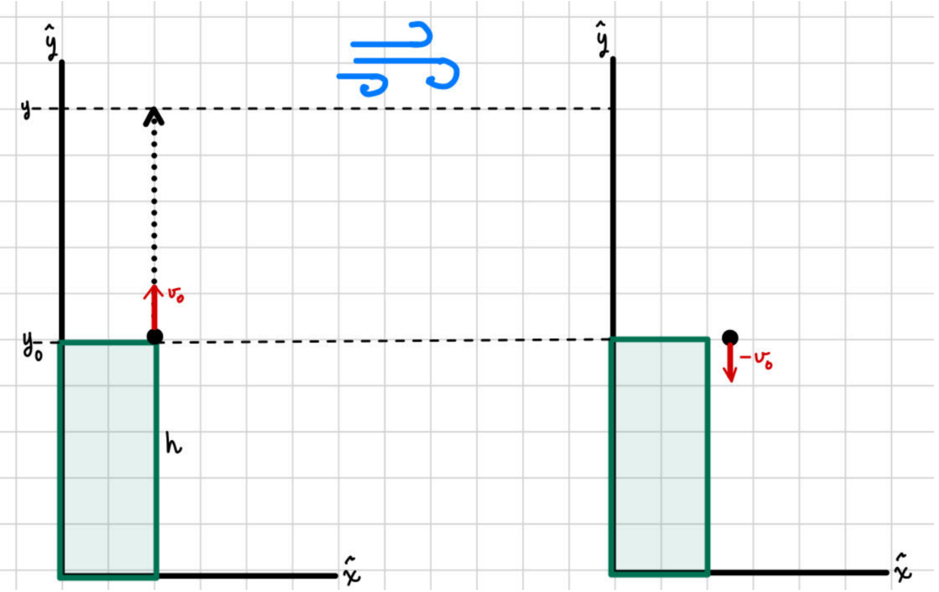
$$y - y_0 = \frac{1}{2}(v_y + \cancel{v_0})t \quad \rightarrow \quad y - h = \frac{1}{2} v_y t \quad \rightarrow h = y - \frac{1}{2} v_y t = 10m + \frac{1}{2}(162.5m/s)(100s) = 8135\ m$$



Example: Taking Advantage of Symmetry

In vertical motion under constant acceleration, the path up mirrors the path down. A projectile's speed when it returns to a previous height is equal in magnitude to its launch speed — just in the opposite direction.

A cannonball is fired straight upward from the edge of a cliff of height $h = 100\ m$ above the ground. It rises vertically, reaches a maximum height, and as it begins to descend, a gust of wind nudges it slightly sideways, causing it to fall past the cliff and land on the ground below. Neglect air resistance and assume $g = -9.81\ m/s^2$ and the cannonball is launched at $v_0 = 7000\ m/s$. How fast is the cannonball moving when it hits the ground?



Reasoning: *We could find the maximum height of the ball, but this is like the rocket problem. The problem only asks for the velocity of the ball just before it hits the ground. The velocity of the cannonball on the return trip, when it passes $y_0 = h = 100\ m$ will be equal to its initial velocity, but downward.*

Step One: Draw a diagram

Step Two: List your known variables: $y_0 = h = 100m,\ v_0 = 7000\frac{m}{s},\ g = -9.81\frac{m}{s^2},\ y_f = 0, v = 0$

Step Three: List your unknown variables: y, v_f

Step Four: write out the Kinematic Equations simplify and solve. Replace v_0 with $-v_0$.

$$v_y = v_0 + a_y t \quad \rightarrow \quad v_f = -v_0 + gt \quad \rightarrow \quad v_f = -7000\ m/s - (9.81\ m/s^2)(0.029\ s) = -7000.28\ m/s$$

$$y - y_0 = v_0 t + \frac{1}{2}a_y t^2 \quad \rightarrow \quad y - y_0 = -v_0 t + \frac{1}{2}gt^2$$

$$y - y_0 = \frac{1}{2}(v_y + v_0)t \quad \rightarrow \quad y - y_0 = -\frac{1}{2}v_0 t \rightarrow t = \frac{-2(y - y_0)}{v_0} = \frac{-2(0 - h)}{v_0} = \frac{2h}{v_0} = \frac{2(100\ m)}{7000\ m/s} = 0.029\ s$$