Kinematics in Two Dimensions

Chapter 3

Introduction

2D Kinematics: *Motion with both x and y components — direction and angle matter*. Examples:

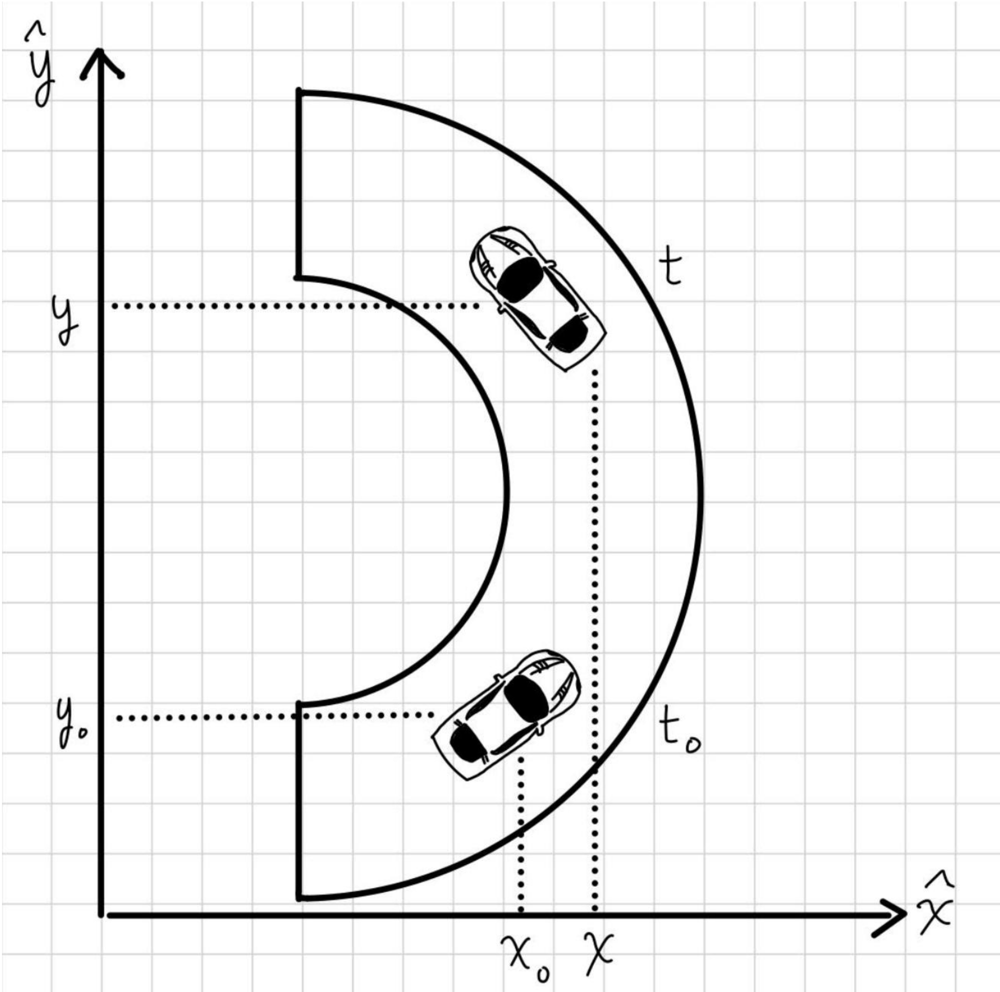
- **Projectile motion** (e.g., a football kicked at an angle)
- **Thrown object on Earth** (e.g., rock tossed into a lake)
- **Cannonball fired at an angle**
- **Car going around a bend or curve**
- **Object sliding off a table** (horizontal + vertical motion)
- **Bird flying diagonally or in an arc**
- **Skater launching off a ramp**
- **A drone moving at an angle in the sky**
- **A boat crossing a river with current (relative velocity)**
- **Satellite in elliptical orbit** (non-uniform curved path)

💡 *Requires breaking motion into x and y components and applying kinematic equations to both.*

Fortunately, we already have the tools to analyze scenarios of this type!

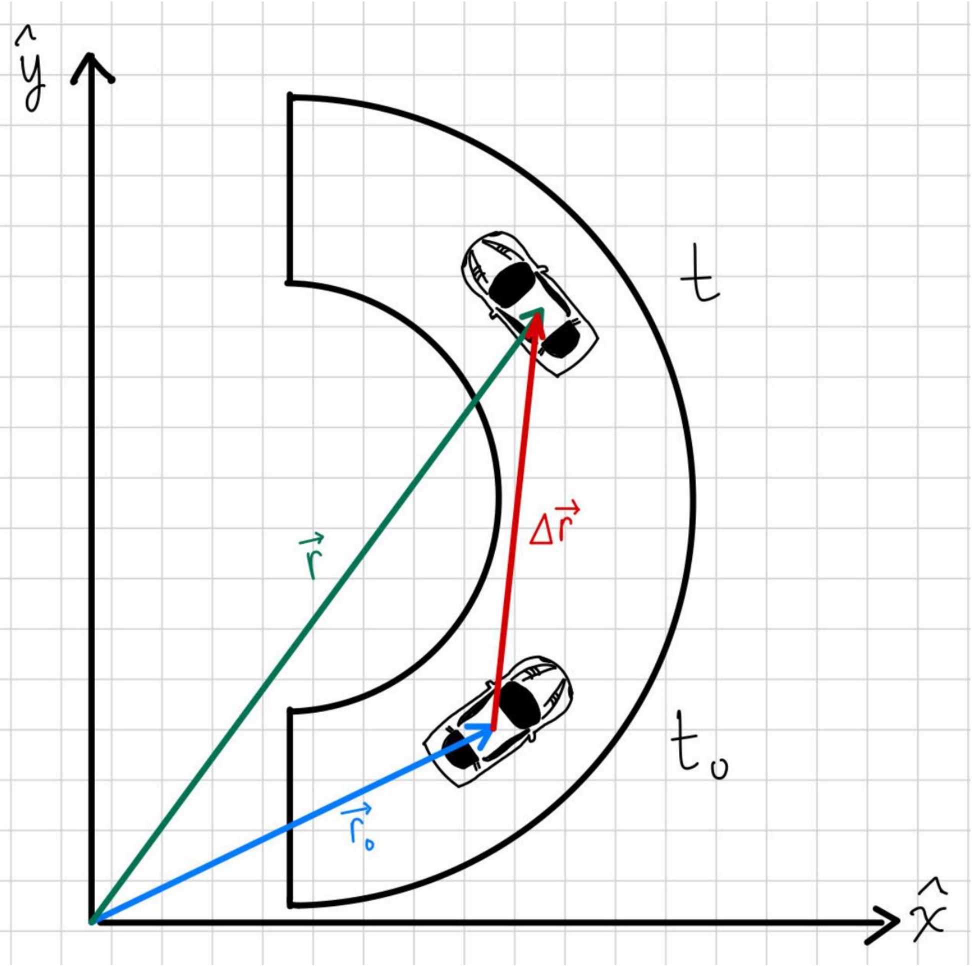
Displacement in 2D

- Consider a race car on a track at two different points in time, t_0, and some time later, t.
- At t_0 the car can be assigned (x_0, y_0).
- At t the car can be assigned (x, y)
- We want to describe its position from the origin at all times – how do we do that?
- Vectors!

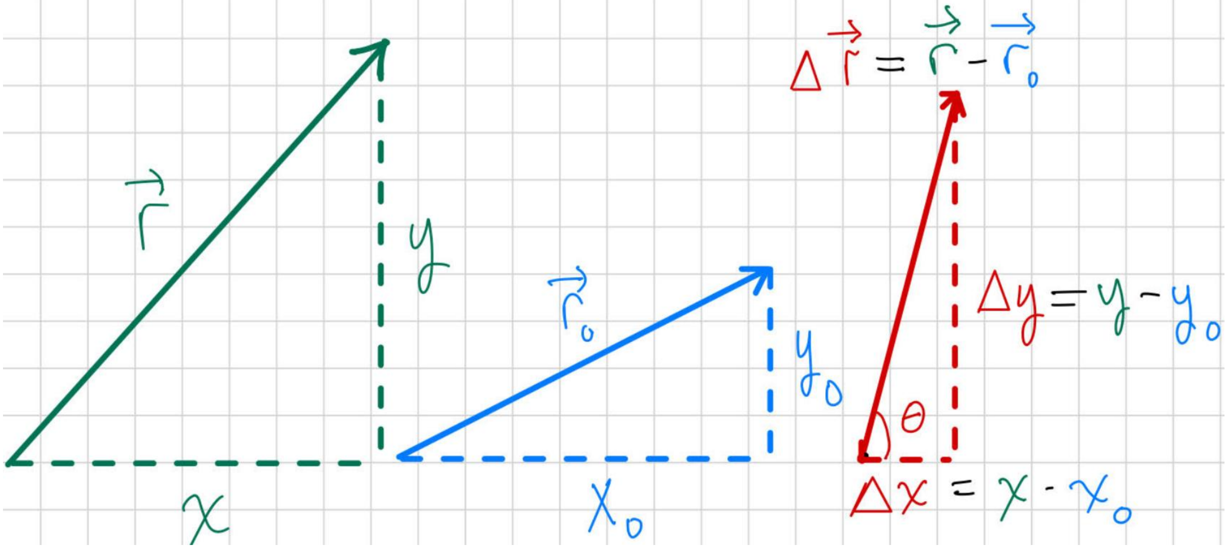


Displacement in 2D 1/3

- Assign vector $\vec{r}_o$ to run from the origin to (x_o, y_o).
- Assign vector $\vec{r}$ to run from the origin to (x, y).
- The difference between these vectors,
  - $\Delta\vec{r} = \vec{r} - \vec{r}_0$



Displacement in 2D 2/3



$$\Delta\vec{r} = \vec{r} - \vec{r}_o$$

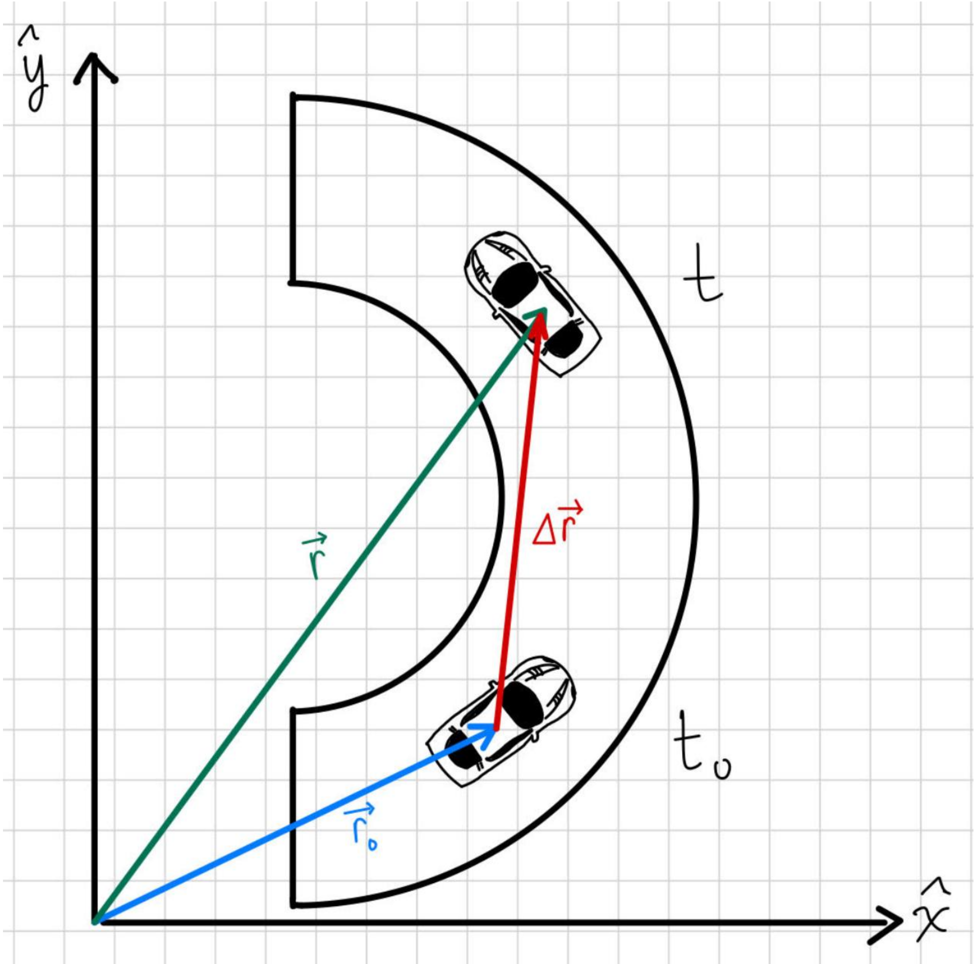
$$\Delta r = \sqrt{\Delta x + \Delta y}$$

$$\Delta x = x - x_0 \qquad \Delta y = y - y_0$$

$$\theta = \tan^{-1}\frac{\Delta y}{\Delta x}$$

Difference vector has:

- Magnitude
- Components
- Angle



Example: Displacement in 2D 3/3

Problem Statement: A racecar is navigating a curved section of a track. At time $t_0 = 0\ s$, the care is located at position $(x_0, y_0) = (40\ m, 30\ m)$. After 2.5 seconds, it reaches a new position at $(x, y) = (45\ m, 60\ m)$.

Question: Determine the displacement vector $\Delta\vec{r}$ between these two positions.

$$\Delta\vec{r} = \vec{r} - \vec{r}_0 = (x - x_0, y - y_0) = (45 - 40, 60 - 30)m$$

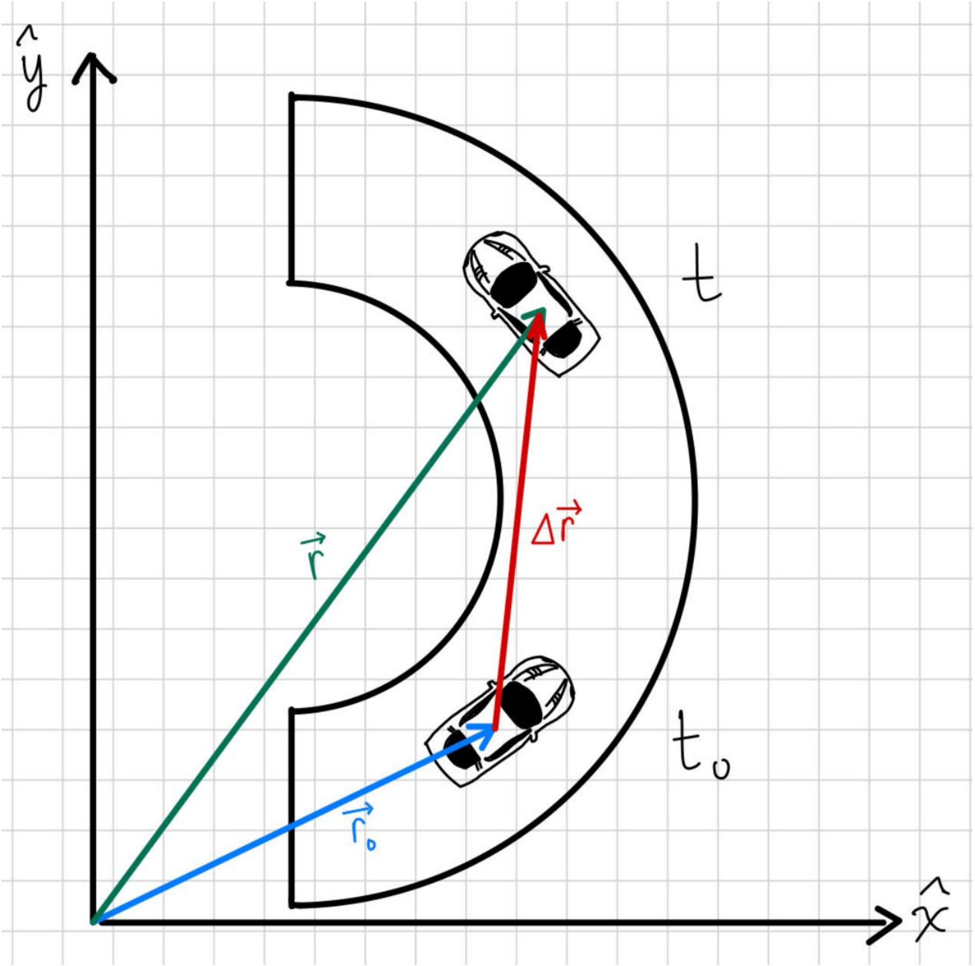
$$= \langle 5\ m, 30\ m \rangle$$

The bracket notation indicates that these are components rather than coordinates.

$$\Delta r = \sqrt{(\Delta x)^2 + (\Delta y)\text{^}2} = \sqrt{(5\ m)^2 + (30\ m)^2} = 30.4\ m$$

$$\theta = \tan^{-1}\frac{\Delta x}{\Delta y} = \tan^{-1}\frac{30}{5} = 80.5^0$$

This angle is measure counterclockwise from $\hat{x}$.



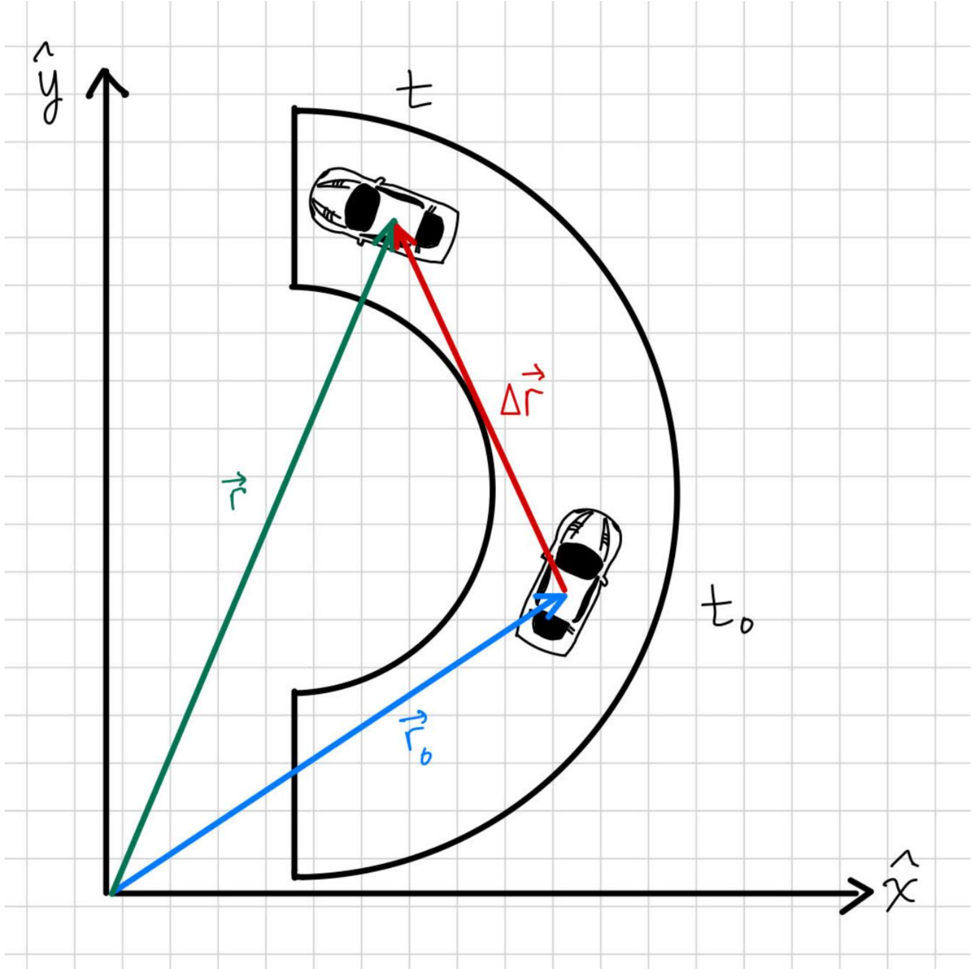
Team Activity: Concept Check 3.1

In the diagram, notice that the displacement vector $\Delta\vec{r}$ has a negative x-component: $\Delta x < 0$.

If we compute the angle using

$$\theta = \tan^{-1}\frac{\Delta y}{\Delta x},$$

will the resulting angle be measured from the negative or positive $\hat{x}$ axis?

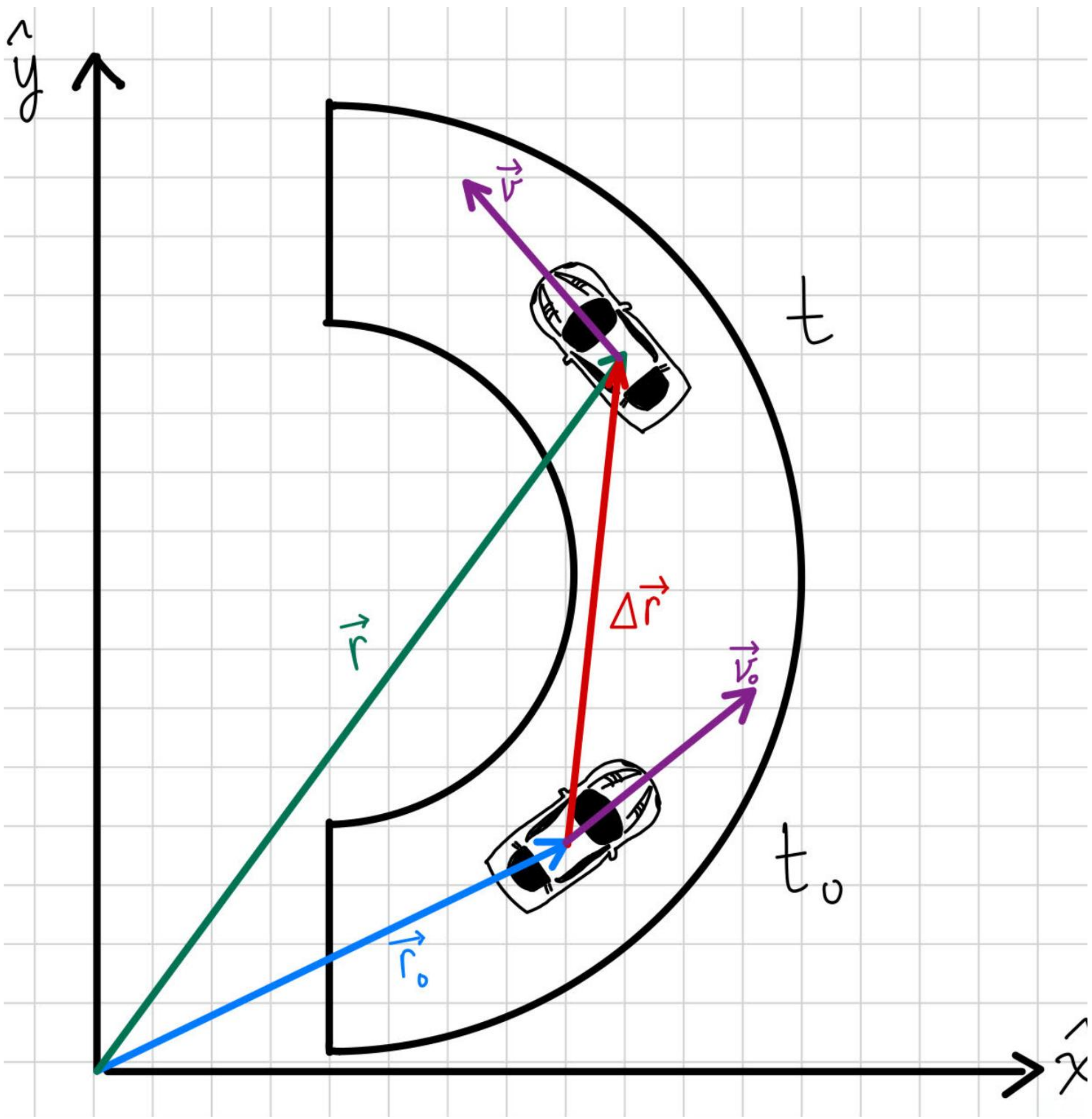


Average Velocity in 2D

- With a displacement vector in 2D, we can compute the average velocity in 2D using,

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t}$$

- As with the 1D case, this doesn't tell us how fast the car was going at a specific instant in time. For that, we need instantaneous velocity (shown in purple).



Instantaneous Velocity in 2D

- The figure shows that the instantaneous velocity vector, is tangent to the path of the car.
- Like any vector, it can be written in component form,

$$v_{0x} = v_0 \cos\theta \qquad v_{0y} = v_0 \sin\theta$$

Sanity Check:

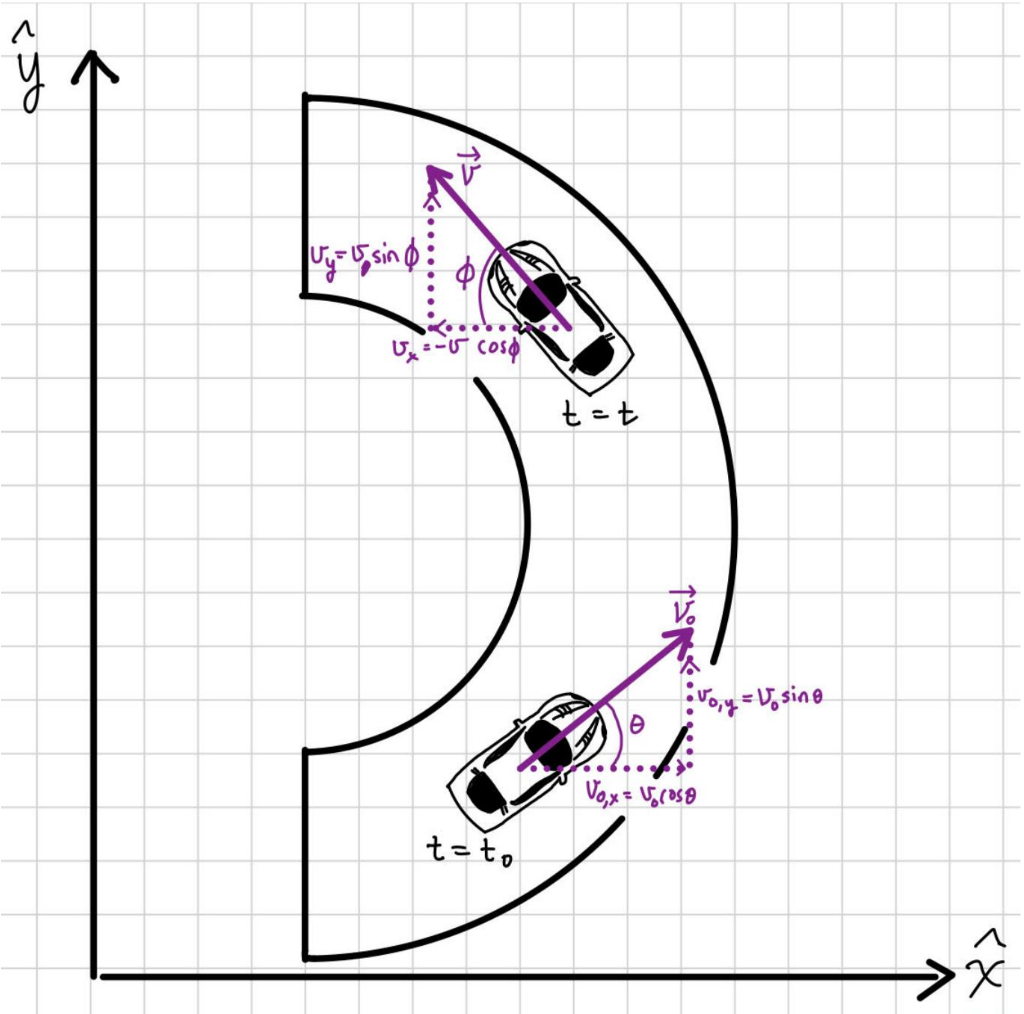
$$v_0 = \sqrt{v_0^2 \cos^2\theta + v_0^2 \sin^2\theta} = v_0\sqrt{\sin^2\theta + \cos^2\theta} = v_0$$

$$v_x = v\cos\phi \qquad v_y = v\sin\phi$$

Sanity Check:

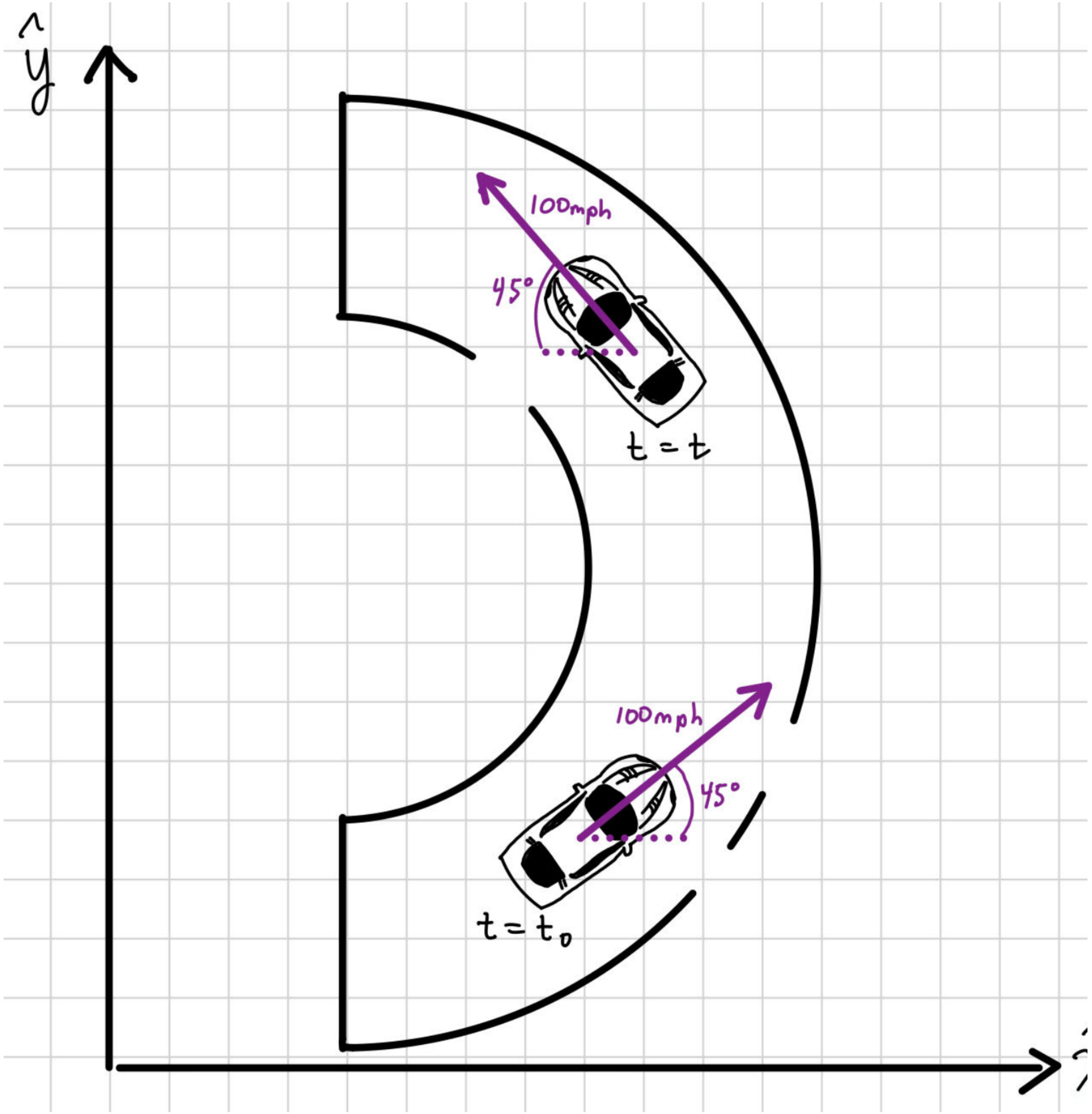
$$v = \sqrt{v^2 \cos^2\phi + v^2 \sin^2\phi} = v\sqrt{\sin^2\phi + \cos^2\phi} = v$$

Note: we used the identity $sin^2\,\alpha + cos^2\,\alpha = 1$



Team Activity: Concept Check 3.2

In the diagram, are the instantaneous velocities at t_0 and t equal?



Equations of Kinematics in 2D

The equation of Kinematics can be written in vector form as,

$$\vec{r}(t) = \vec{r}_o + \vec{v}_o t + \frac{1}{2}\vec{a}t^2 \qquad \vec{v}(t) = \vec{v}_o + \vec{a}t^2$$

In component form,

$$x = x_o + v_{ox}t + \frac{1}{2}a_x t^2 \qquad v_x = v_{ox} + a_x t \qquad v_x^2 = v_{ox}^2 + 2a_x(x - x_o)$$

$$y = y_o + v_{oy}t + \frac{1}{2}a_y t^2 \qquad v_y = v_{oy} + a_y t \qquad v_y^2 = v_{oy}^2 + 2a_y(y - y_o)$$

Where the third equation is created from the first two.

You already know how to use either the x-component or y-component equations independently. In 2D you use them together! After solving, you can just recombine,

$$\vec{r}(t) = x(t)\hat{x} + y(t)\hat{y} \qquad \vec{v}(t) = v_x\hat{x} + v_y\hat{y}$$

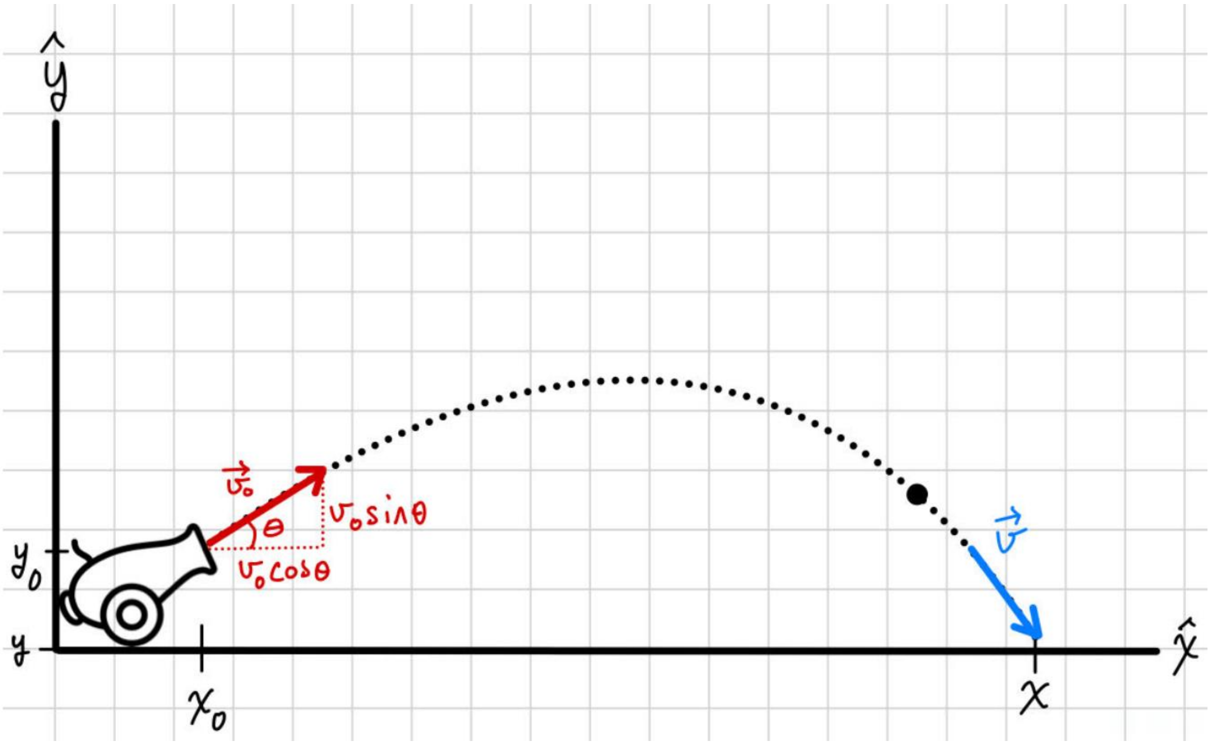
Projectile Motion

To use them together you connect them through geometry! Specifically, the angle θ!

What is the acceleration in x? $a_x = 0$. In y? $a_y = -g$.

Generic Projectile Motion Equations,

- $x = x_o + v_o \cos\theta\, t$
- $v_x = v_o \cos\theta$
- $v_x^2 = v_o^2 \cos^2\theta$
- $y = y_o + v_o \sin\theta\, t - \frac{1}{2} g t^2$
- $v_y = v_o \sin\theta - gt$
- $v_y^2 = v_o^2 \sin^2\theta - 2g\,(y - y_o)$



Notice in this specific example $y = 0$ which means that our final velocity at that point, just before hitting the ground, will be greater than v_0 in terms of magnitude.

Projectile Motion: Max Height

How do we find the maximum height? Notice at the top of the curve, just before the ball starts to descend, $v_y = 0$.

$$v_y^2 = v_o^2 \sin^2\theta - 2g\,(y_{max} - y_o) = 0$$

$$\boxed{y_{max} = y_0 + \frac{v_0^2 \sin^2\theta}{2g}}$$

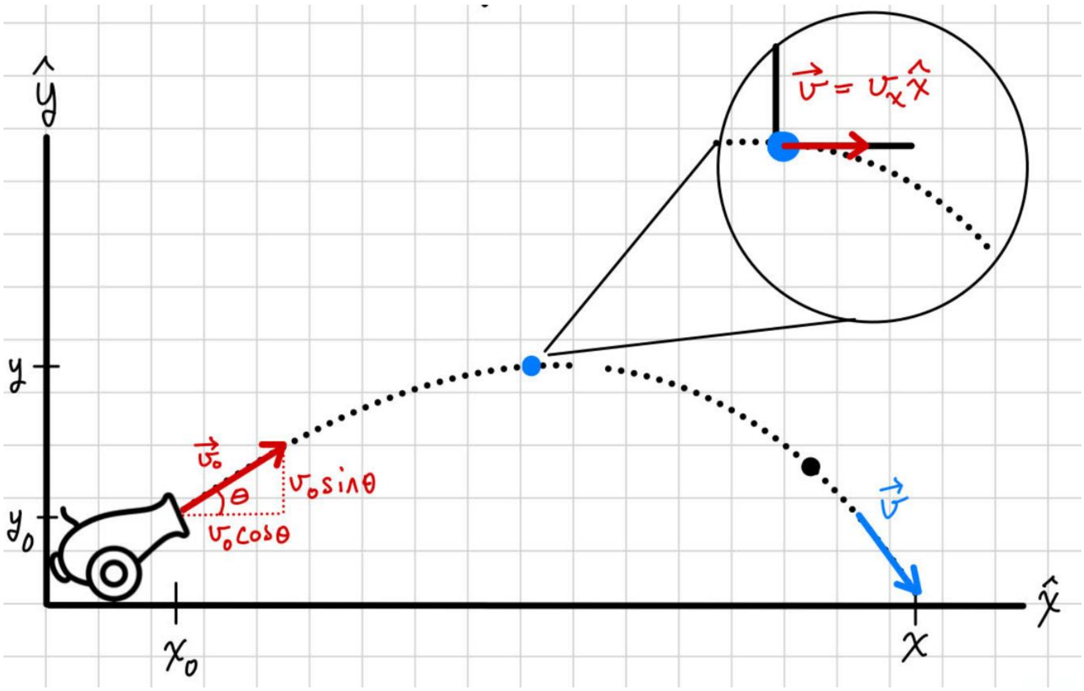
Now we can find the time to reach y_{max} :

$$v_y = v_o \sin\theta - gt = 0 \quad \rightarrow \quad \boxed{t_{max} = \frac{v_o \sin\theta}{g}}$$

And what the x location will be,

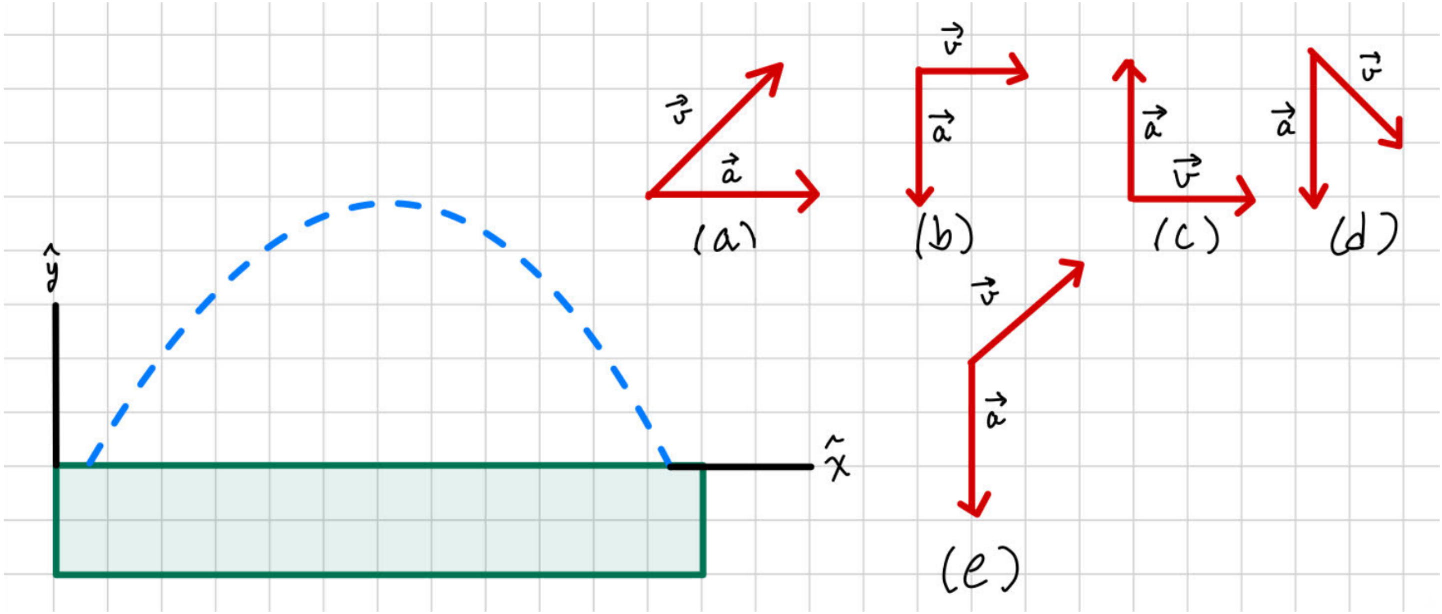
$$x = x_o + v_o \cos\theta\, t = x_o + v_0 \cos\theta \left(\frac{v_o \sin\theta}{g}\right)$$

$$\boxed{x_{max} = x_0 + \frac{v_o^2}{g}\sin\theta cos\theta}$$

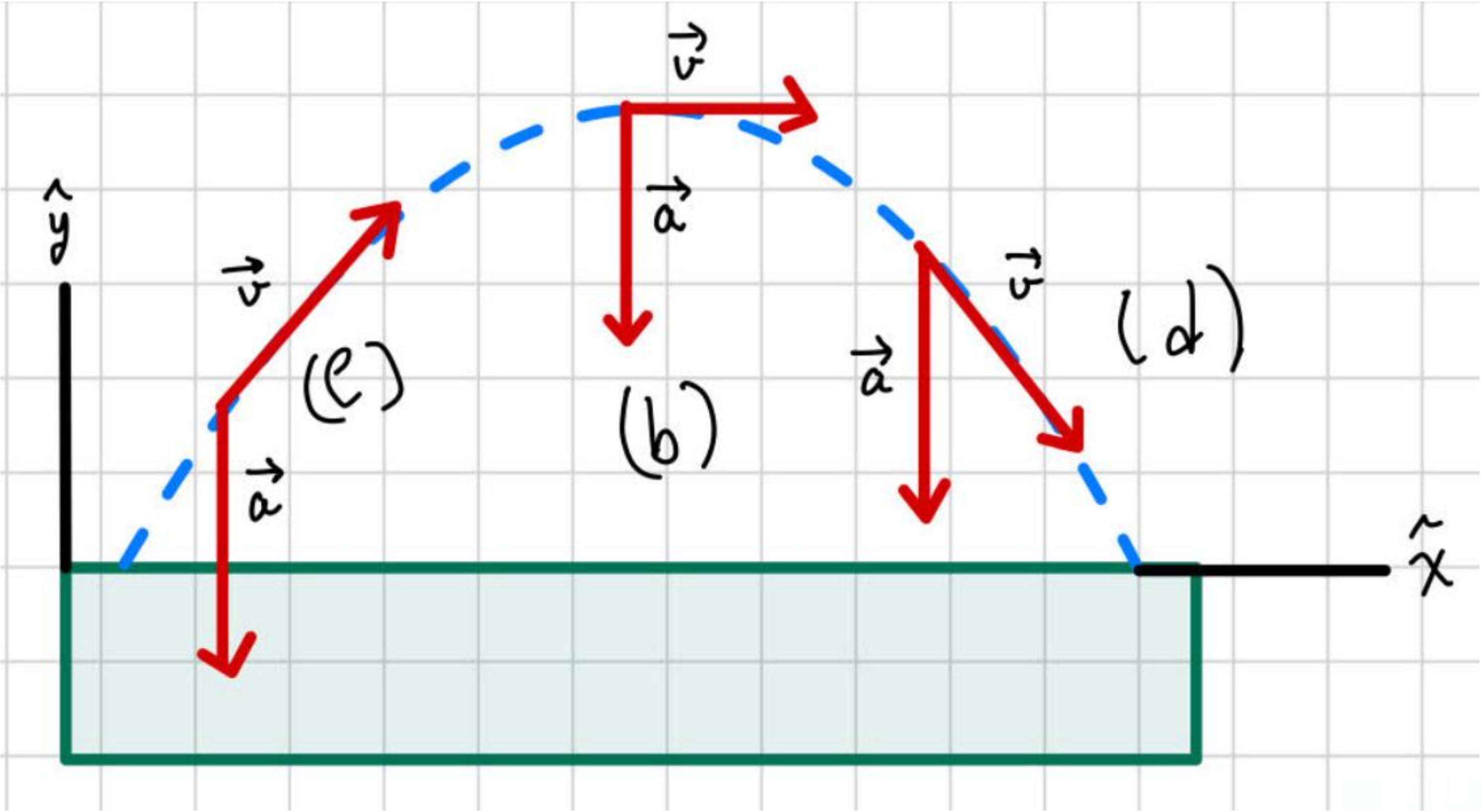


Team Activity: Concept Check 3.3

A projectile is fired into the air; it follows a parabolic path. There is no air resistance. At any instant, the projectile has a velocity, $\vec{v}$, and an acceleration, $\vec{a}$. Which on or more of the drawings could represent the directions for $\vec{v}$ and $\vec{a}$ at points on the trajectory? (vectors not drawn to scale)



Team Activity 3.3 Solution



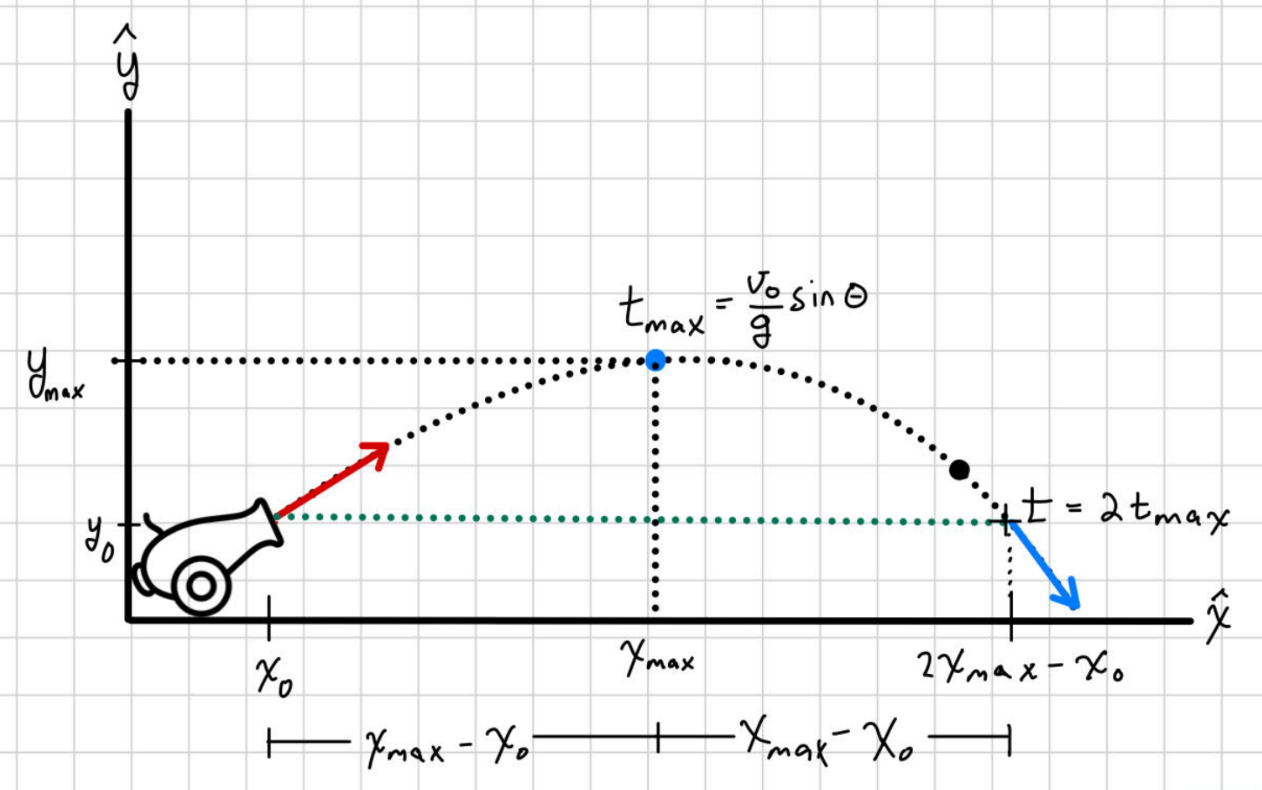
Projectile Motion: Symmetry

- Notice that the ball will pass back through y_0 before it hits the ground – this is the height the cannon ball was fired from.
- Due to the symmetry, we know that the total time of flight from y_0 and back again is

$$t = 2t_{max} = \frac{2v_0}{g}\sin\theta.$$

- Likewise, when the ball returns to a height of y_0, it will be just above

$$2x_{max} - x_0 = \frac{v_o^2}{g}\sin\theta cos\theta$$



Due to this symmetry, these points at maximum height are often referred to as $x_{1/2},\ y_{1/2},\ t_{1/2}$

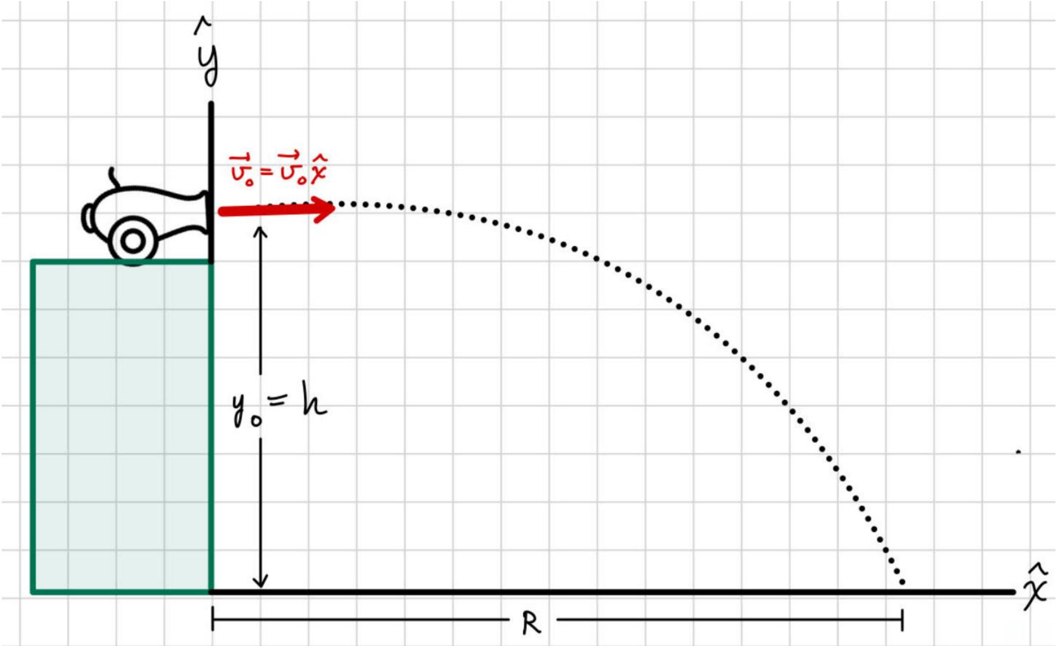
Team Activity: Concept Check 3.4

- We learned that in projectile motion, the **y-component** of velocity decreases during ascent, reaches zero at maximum height, and then increases in the negative direction during descent.
- What about the **x-component** of velocity? Does it change in the same way? Why or why not? What **would** have to be true for the x-velocity to change during flight?

Projectile Motion: Straight Shot

- Do we really need new equations for this? Just use our generic projectile motion equations with $\theta = 0$
- $x = x_o + v_o \cancelto{1}{\cos\theta}\, t \quad \rightarrow \quad x - x_0 = R = v_0 t \quad \rightarrow \quad t = \frac{R}{v_0}$
- $v_x = v_o \cancelto{1}{\cos\theta} = v_o$
- $v_x^2 = v_o^2 \cancelto{1}{\cos^2\theta} = v_0^2$
- $y = y_o + v_o \cancelto{0}{\sin\theta}\, t - \frac{1}{2} g t^2 \rightarrow 0 = h - \frac{1}{2} g t^2 \rightarrow t = \sqrt{\frac{2h}{g}}$
- $v_y = v_o \cancelto{0}{\sin\theta} - gt = -gt$
- $v_y^2 = v_o^2 \cancelto{0}{\sin^2\theta} - 2g\,(y - y_o) = 2gh$
- We now have two expressions for time. Set them equal!

$$\frac{R}{v_o} = \sqrt{\frac{2h}{g}} \quad \rightarrow \quad \boxed{v_0 = R\sqrt{\frac{g}{2h}}}$$



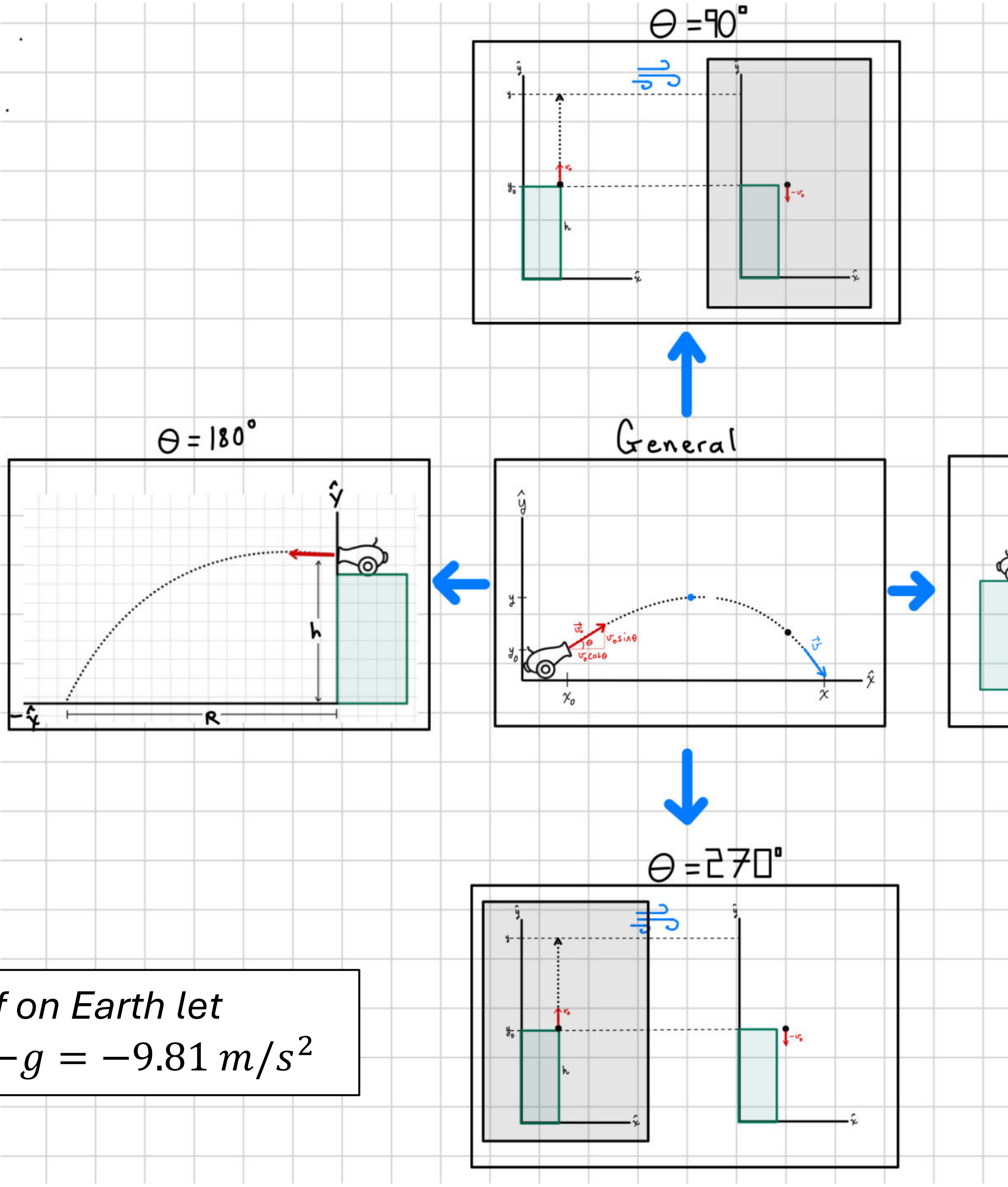
*"We're not learning new physics — we're just changing the **initial conditions**. The math structure is the same. What changes is the story we plug in." – Some guy*

One Structure To Rule Them All!

- We have worked all these different geometries – some in the 1D Kinematics lecture and some in the current one.
- But the general case can handle them all with adjustments to angle – and then the other variables as needed.
- If you understand the general case, you understand all cases!

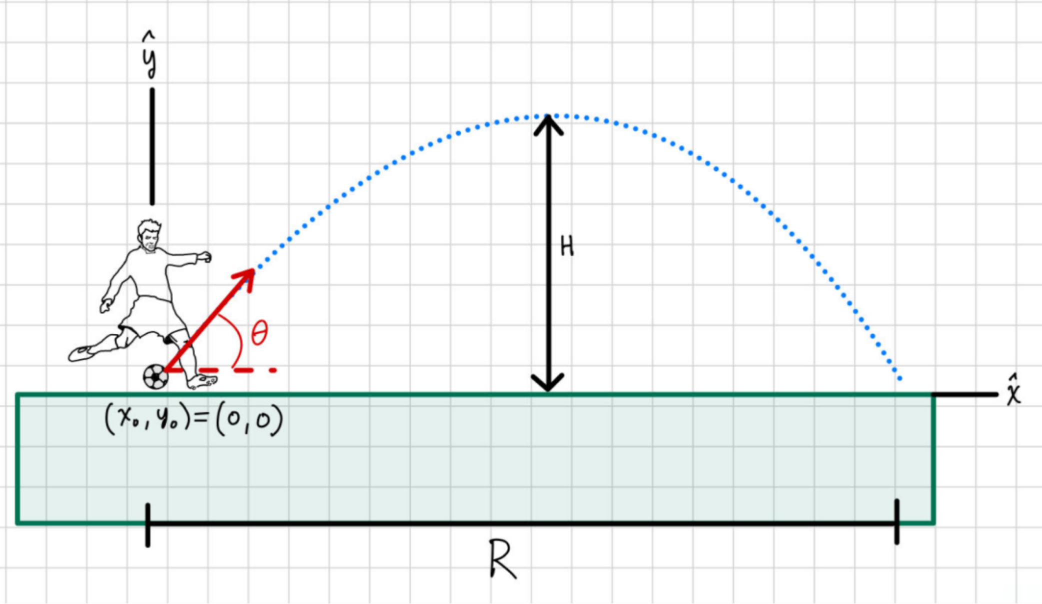
$$x = x_o + v_o \cos\theta\, t$$
$$v_x = v_o \cos\theta$$
$$v_x^2 = v_o^2 \cos^2\theta$$
$$y = y_o + v_o \sin\theta\, t + \frac{1}{2} a_y t^2$$
$$v_y = v_o \sin\theta + a_y t$$
$$v_y^2 = v_o^2 \sin^2\theta + 2a_y\,(y - y_o)$$

If on Earth let
$a_y = -g = -9.81\ m/s^2$



Example: How to Improve Your Soccer

A center forward kicks a soccer ball at an angle of $\theta = 40^0$ above the horizontal axis. The initial speed of the ball is $v_0 = 22\ m/s$. Ignore air resistance and find the maximum height H that the ball attains.



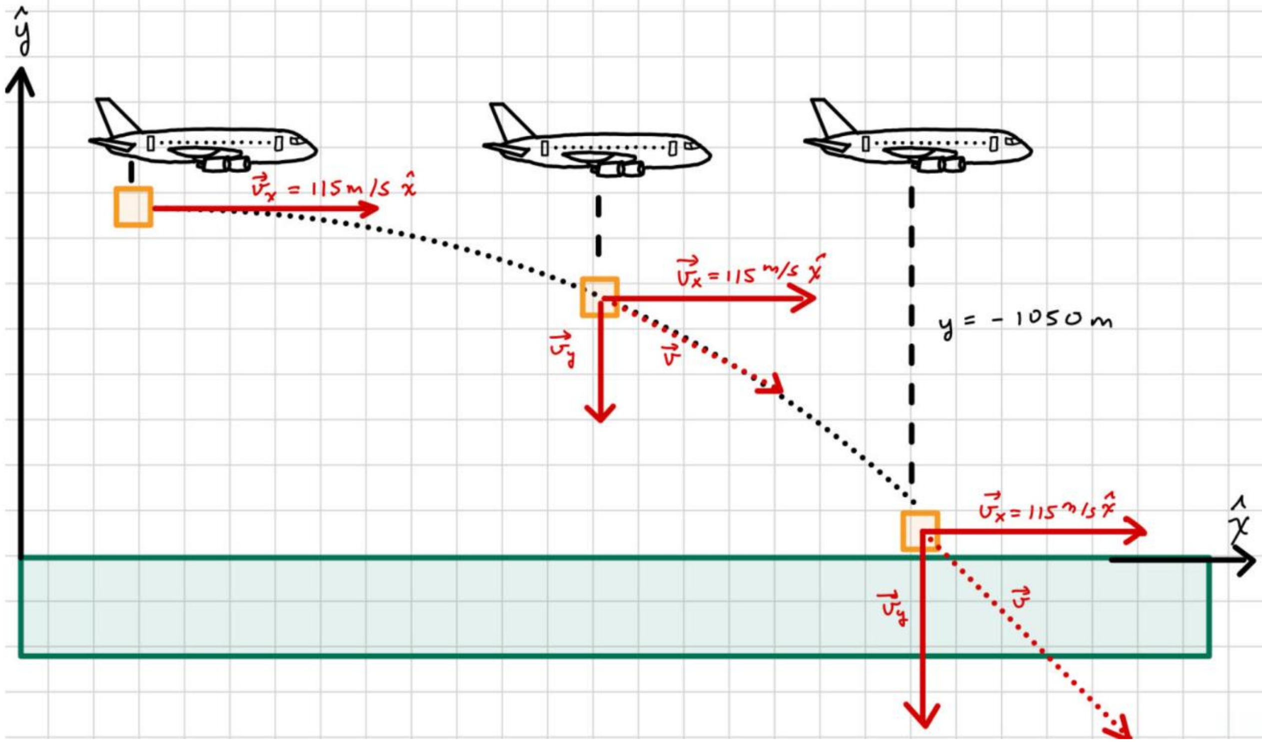
Do we need new physics?

No! We already derived an equation for max height.

$$H = y_0 + \frac{v_0^2 \sin^2\theta}{2g} = 0 + \frac{(22\ m/s)^2 \sin^2 40^0}{2(9.81\ m/s^2)} = 10.2\ m$$

Example: A Falling Care Package

An airplane is flying horizontally with a constant velocity of 115 m/s at an altitude of 1050m. The plane releases a care package that falls to the ground along a curved trajectory. Ignoring air resistance, determine the time required for the package to hit the ground.



Do we need new physics? Nope.

$$y - y_o = v_o \sin\theta\, t - \frac{1}{2}\, gt^2 \rightarrow t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2(1050m)}{9.81m/s^2}}$$

$$t = 14.6\ s$$

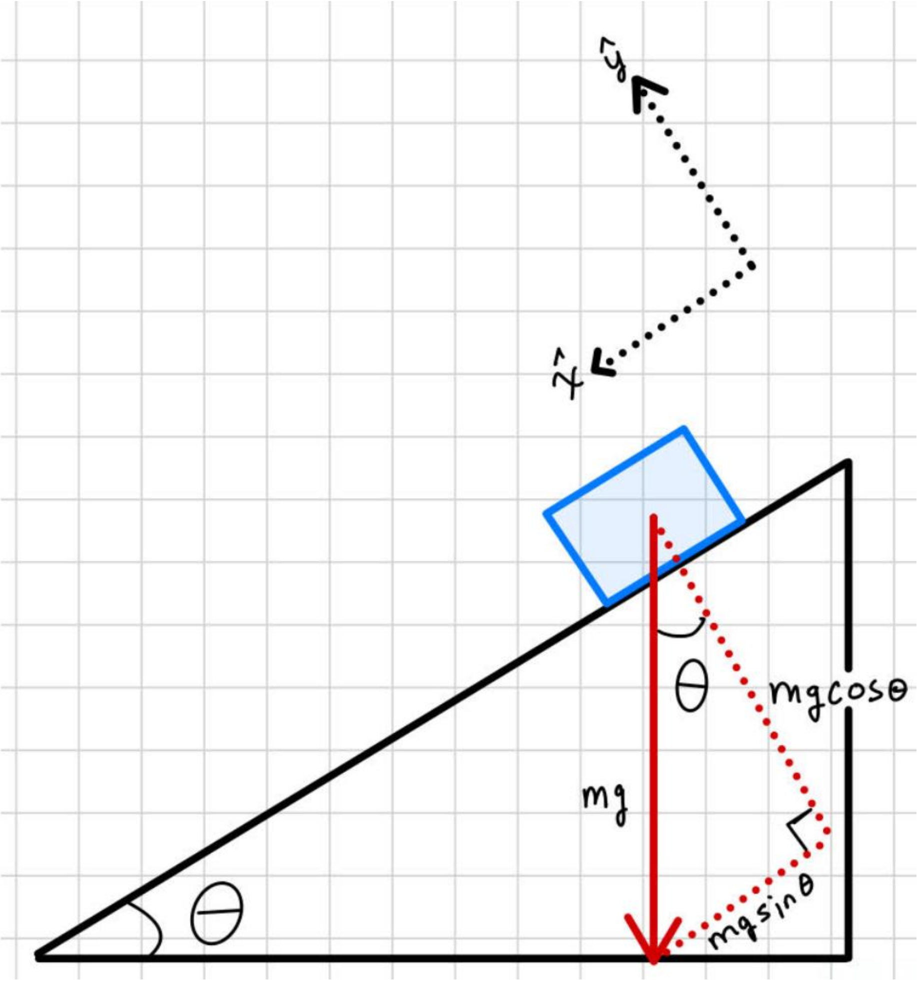
The Sliding Block on an Incline Plane

A block of mass m slides down an incline with an angle θ. Note that the entirety of its motion is determined by gravity in the $\hat{x}$ direction, which is $gsin\theta$. But the y-component, is balanced by the surface the block is resting on, thus it has no motion in the $\hat{y}$ direction. So, we can use our familiar kinematic equations in 2D:

$$x = x_o + v_{ox}t + \frac{1}{2}a_x t^2 \qquad v_x = v_{ox} + a_x t$$

Subbing in the acceleration,

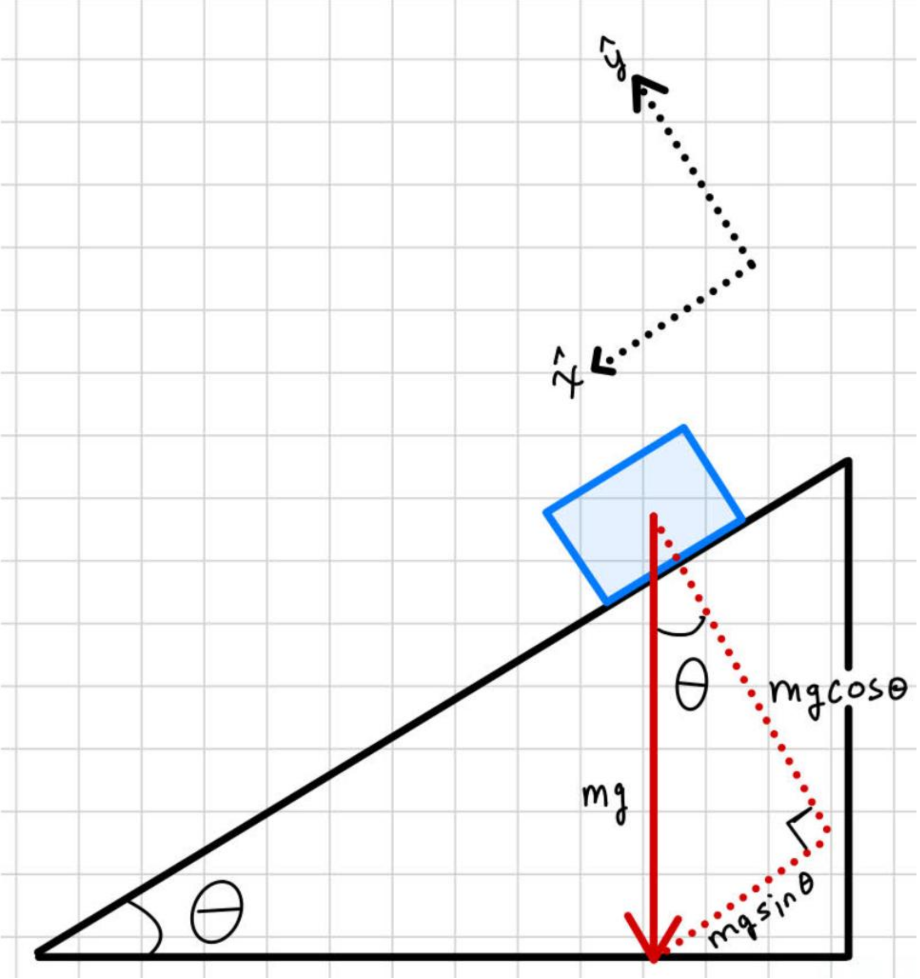
$$x = x_o + v_{ox}t + \frac{1}{2}g\sin\theta\, t^2 \qquad v_x = v_{ox} + g\sin\theta\, t$$



This is the platform we will use to study friction in chapter 4.

Example: The Sliding Block on an Incline Plane

A block of mass m, starts at rest, and then slides down an incline with an angle 45^0. What is the velocity when it passes 20 cm by its starting position?



$$v_{ox} = 0, \qquad x_0 = 0, \qquad g = 9.81\ m/s^2$$

$$x = \frac{1}{2} g \sin\theta\, t^2 \rightarrow \quad t = \sqrt{\frac{2x}{g \sin\theta}}$$

$$v_x = g \sin\theta\, t = g \sin\theta \left(\sqrt{\frac{2x}{g \sin\theta}} \right) = \sqrt{2xg\ sin\theta}$$

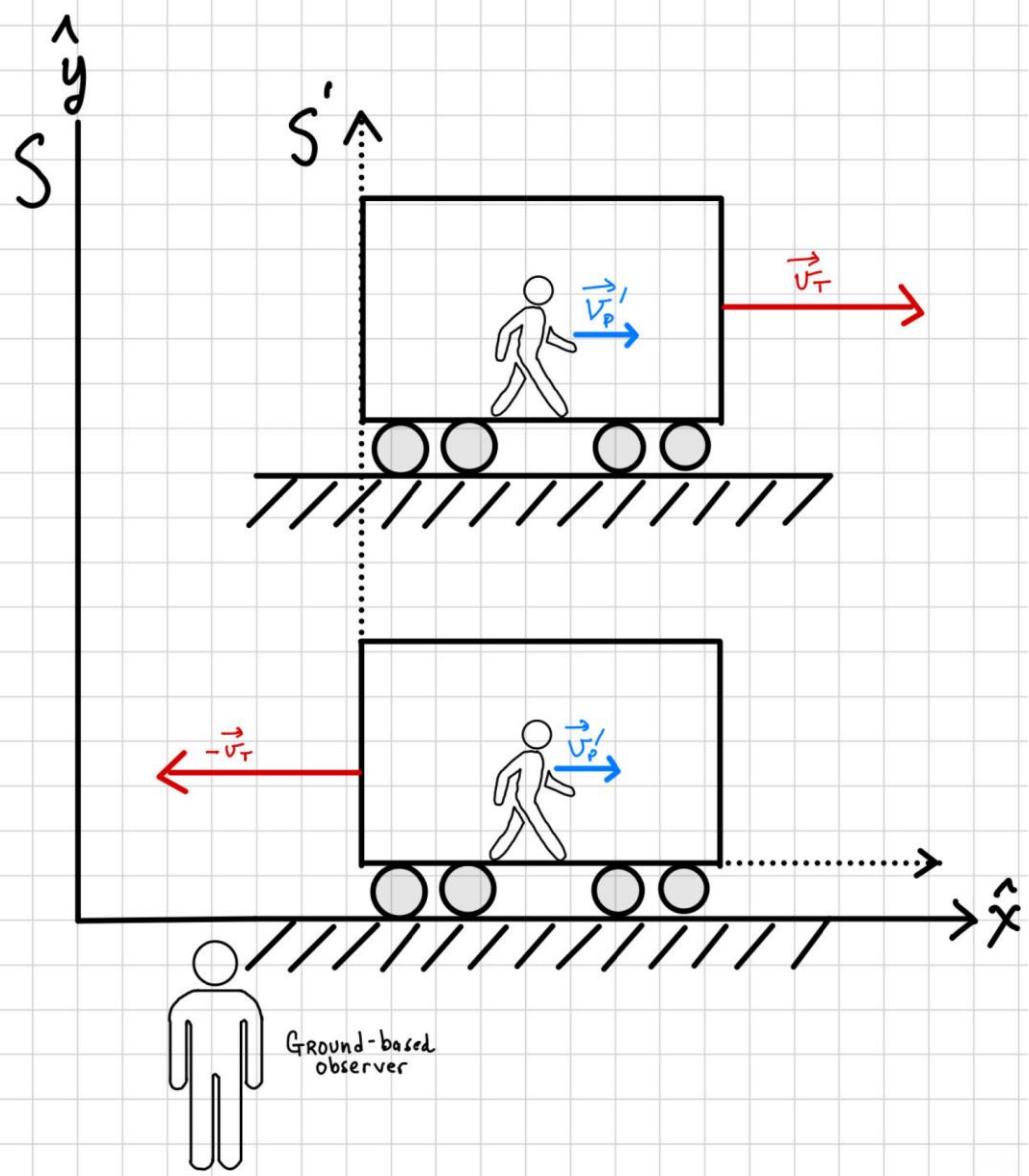
$$= \sqrt{2(0.2\ m)(9.81\ m/s^2)\ (\sin 45^0)}\ = 1.67\ m/s$$

This is the platform we will use to study friction in chapter 4.

Relative Velocity

- Consider a train car with a passenger who is walking with speed $v_p{}'$. The prime indicates that it is measured with respect to the moving train car.
- The train car in the top image is moving toward $\hat{x}$ while the one at the bottom is moving toward $-\hat{x}$.
- We can compute the walking speed of the observer with respect to the ground,

$$\vec{v}_p = \vec{v}_p{}' \pm \vec{v}_T$$



Team Challenge: Relative Velocity

The engine of a boat drives it across a river that is $1800\ m$ wide. The velocity of the boat relative to the water, $\vec{v}_B$, is $4.0\ m/s\,\hat{y}$. The velocity of the water relative to the shore, $\vec{v}_c = 2.0\ \mathrm{m}/s\ \hat{x}$.

a) What is the velocity of the boat relative to the shore, $\vec{v}$? (*Give both magnitude and* θ)

b) How long does it take for the boat to cross the river?

Work this problem as a team. You will use what you learned about relative velocity and kinematics to solve it. Show me your solution, as a team, before you leave.