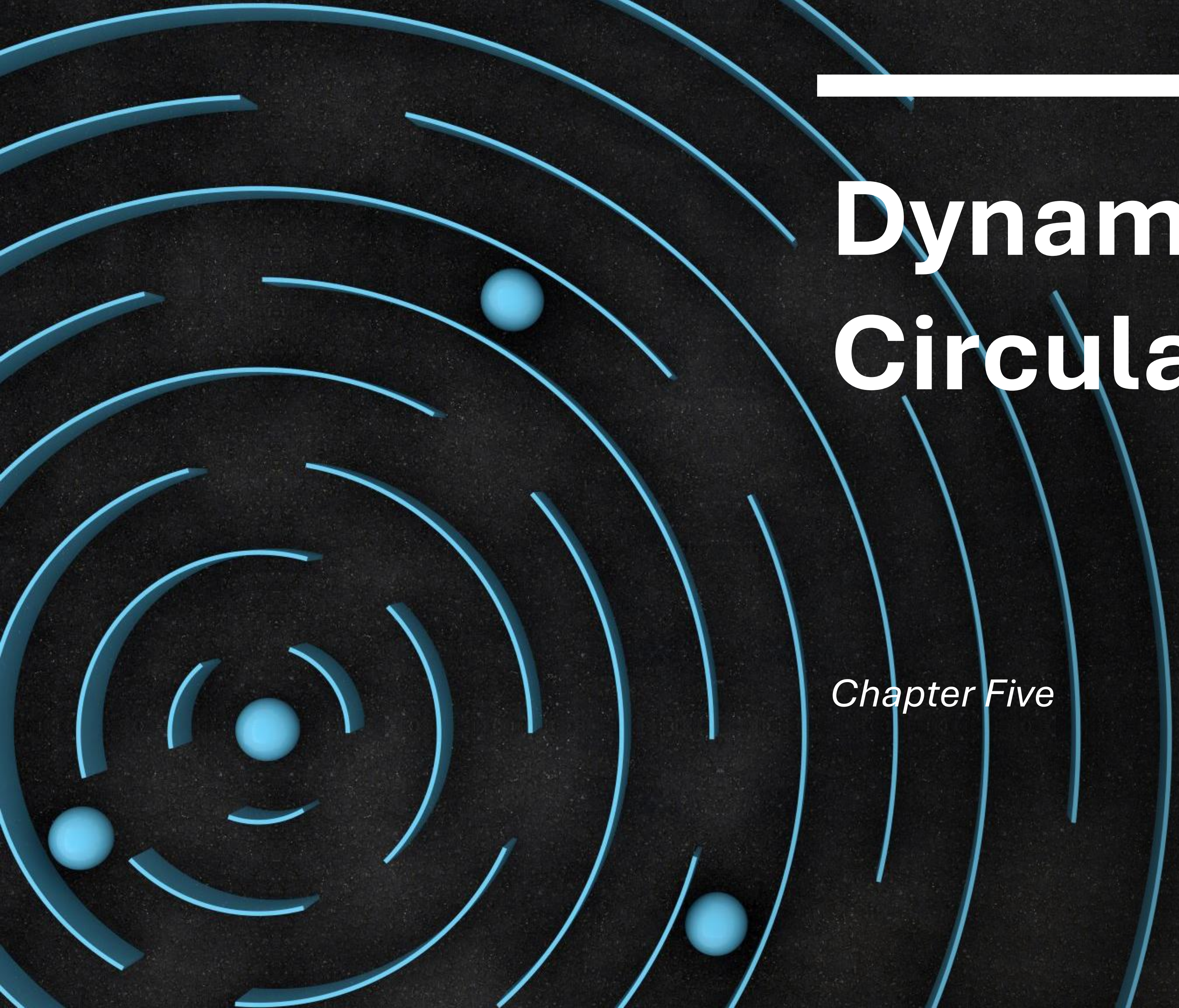
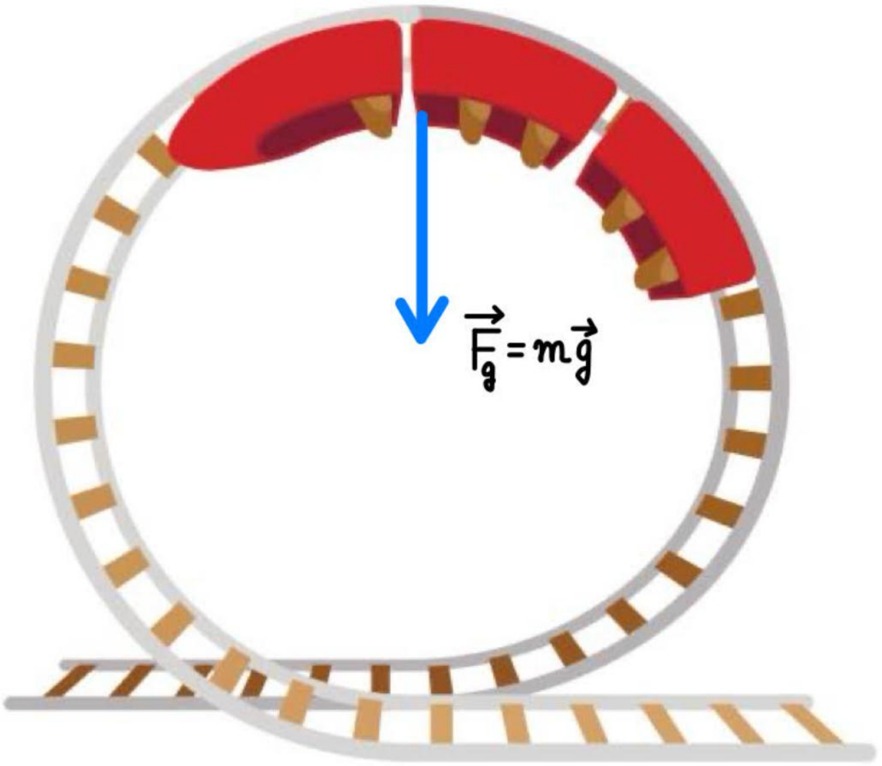


Dynamics of Circular Motion

Chapter Five

Uniform Circular Motion



You’ve probably seen this or maybe felt it yourself. You’re in a roller coaster, going fast and at a constant velocity, and then suddenly you are upside down. But weirdly, you don’t fall. You don’t dangle from your seatbelt. You feel like you're pressed into your seat, even though you are upside down. Why? Gravity is still pulling down. That hasn’t changed. But something else must be happening. Something is keeping you in your seat.

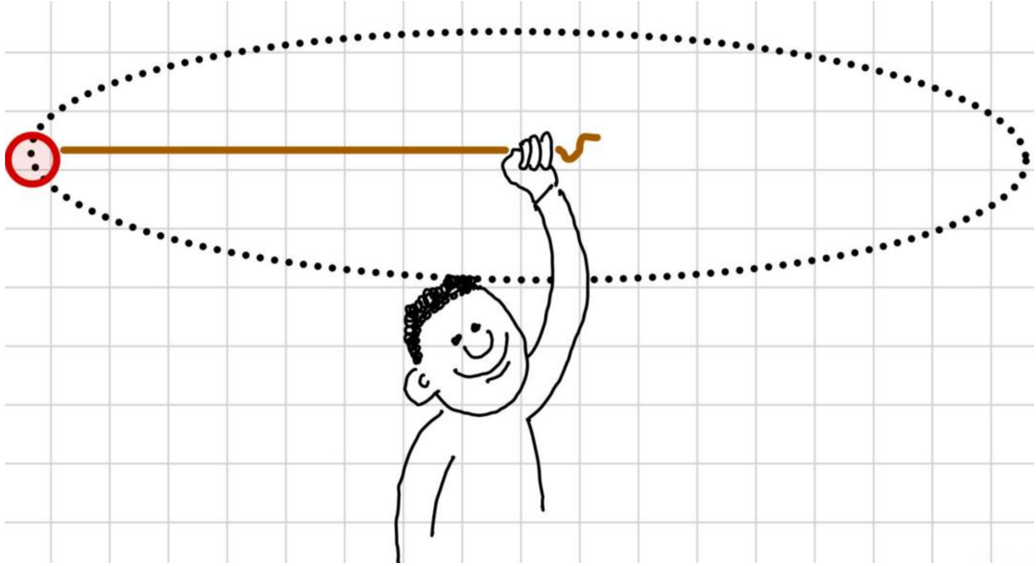
Any ideas?

That’s the question that will lead us into today’s topic - **Uniform Circular Motion, *which is the motion of an object traveling at a constant (uniform) speed on a circular path.***

Team Activity: Concept Question 5.1

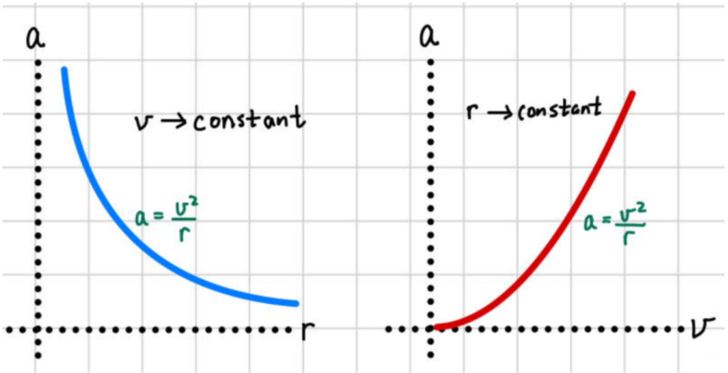
Gus is easily entertained. He spends hours in a field playing with a ball on a string. He spins the ball in the air in uniform circular motion – meaning that it moves at a constant speed along a circular path. He begins to wonder:

Is the acceleration zero, constant, or is it changing?



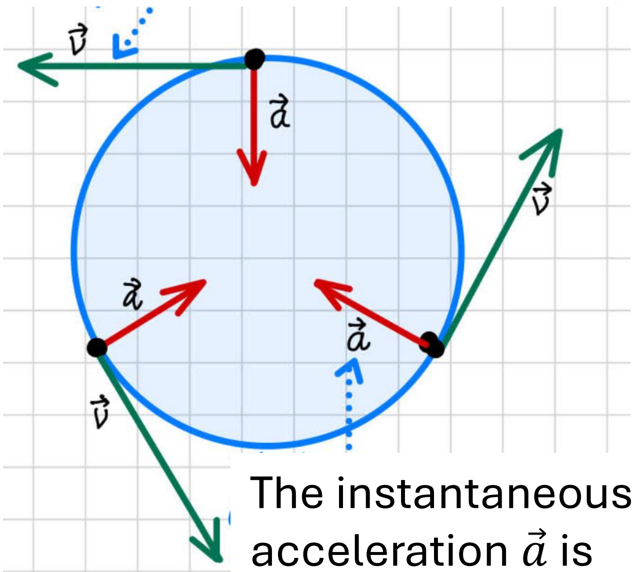
Centripetal Acceleration

- While the speed is constant, recall that the instantaneous velocity, $\vec{v}$, is a vector, and thus if it changes direction, the velocity is not constant – only its magnitude is.
- if $\vec{v}$ is not constant, there must be an instantaneous acceleration, $\vec{a}$.
- In fact, there is an acceleration at every point in the motion, directed toward the center of the circle called ***centripetal acceleration***,



$$\vec{a} = \frac{v^2}{r}\hat{r}$$

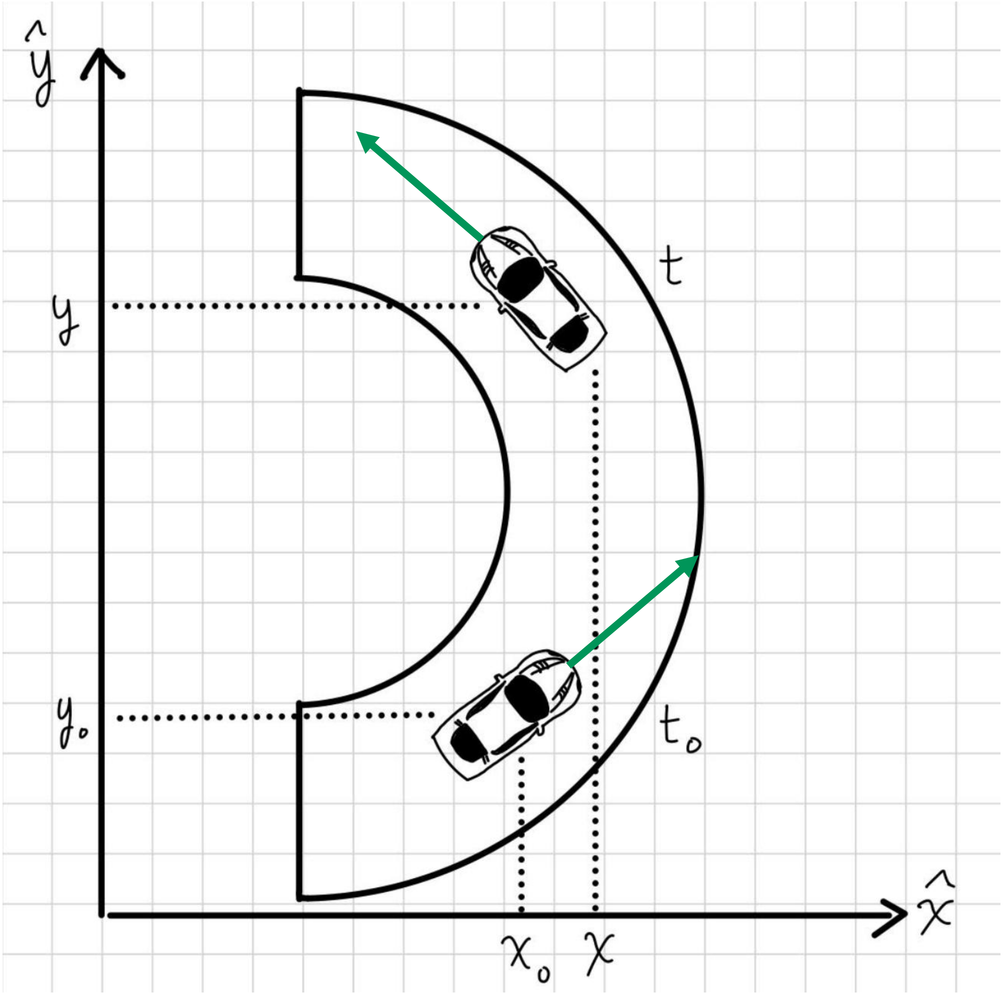
The instantaneous velocity $\vec{v}$ is tangent to the circle at all points.

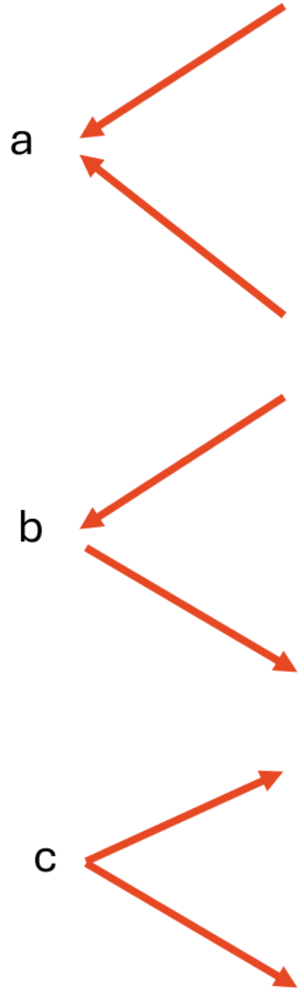


The instantaneous acceleration $\vec{a}$ is directed toward the center of the circle at all points.

Team Activity: Concept Check 5.2

A car is turning a tight corner at a constant speed. A top view of the motion is shown. The velocity vector at t_0 points northeast, and at t points northwest. Which represents the acceleration direction at the two times?





Period, Frequency, and Speed

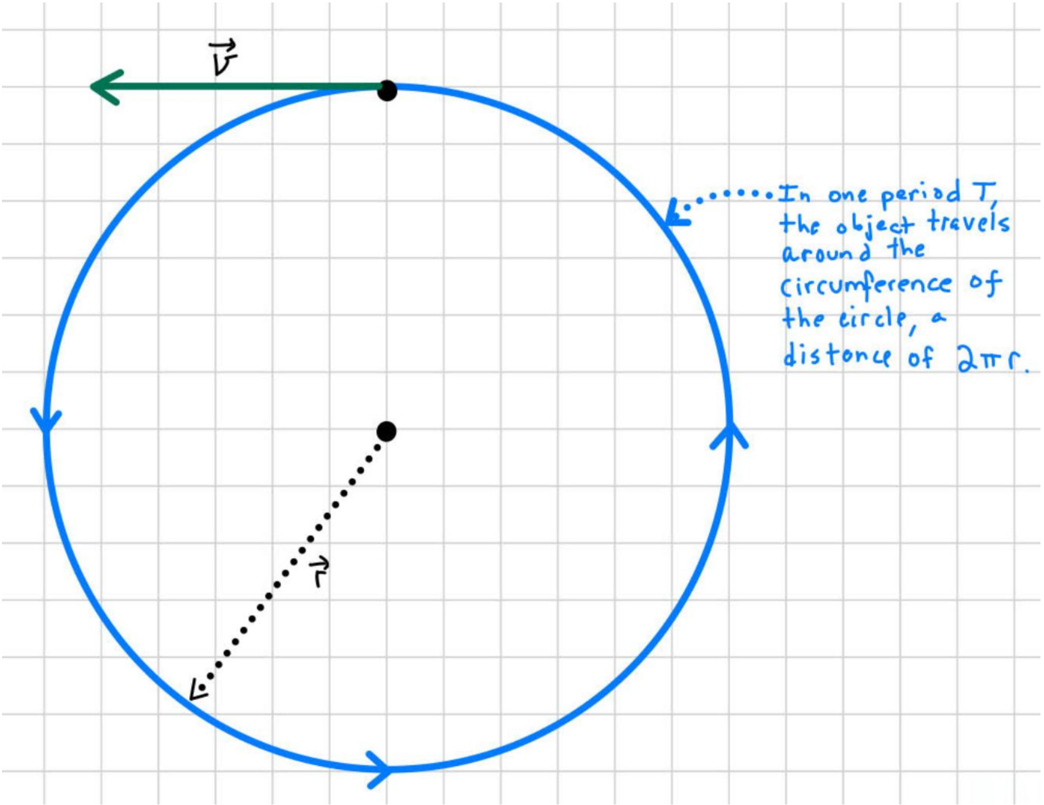
- When a mass moves with uniform circular motion and completes one full circle, the next circle it makes is just a repeat of the first – so the motion is ***periodic***.
- The time interval it takes an object to go around a circle one time, completing one ***revolution*** (rev), is called the ***period*** of motion and is represented by T and sometimes τ. We will use T.
- The ***frequency*** of circular motion is the number of revolutions per second $f = \frac{1}{T}$ and has units $Hz = s^{-1}$.
- The figure shows an object moving at constant speed in a circular path of radius r. We know the time for one revolution – one period T –and we know the distance around the circle, the circumference, $C = 2\pi r$. So we can write the speed, distance over time, as,

$$v = \frac{2\pi r}{T} = 2\pi f r = \omega r$$

- Now we can define centripetal acceleration in terms of period or frequency as,

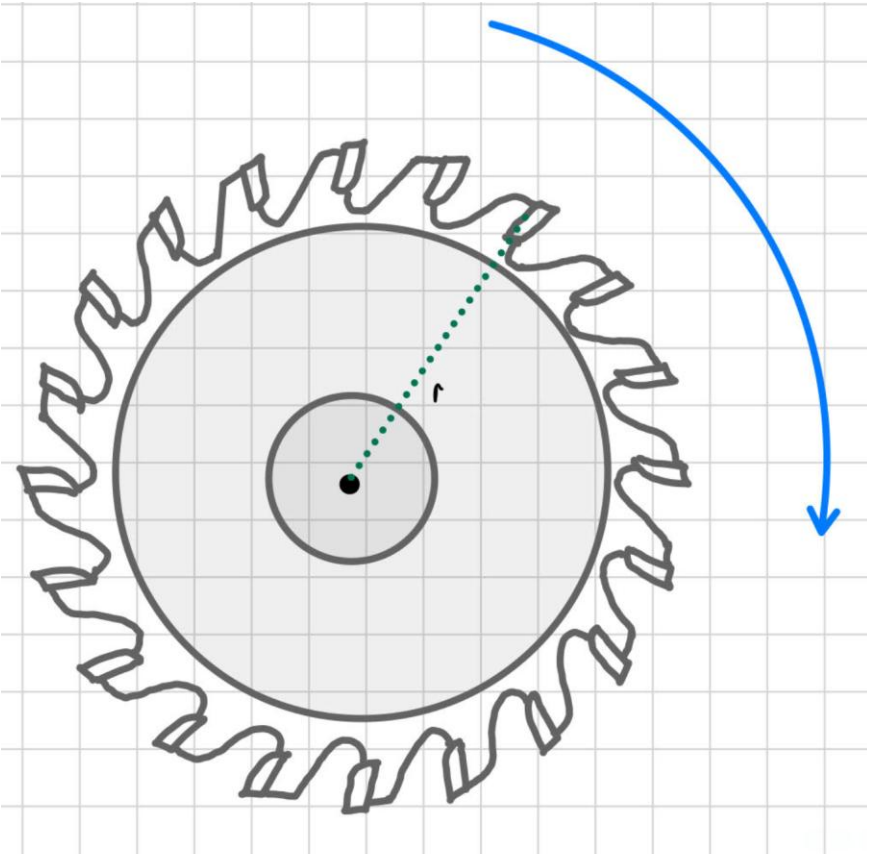
$$a = \frac{v^2}{r} = (2\pi f)^2 r = \left(\frac{2\pi}{T}\right)^2 r = \omega^2 r$$

- Here, we have used $\omega = 2\pi r = 2\pi/T$, which is known as the ***angular frequency***. Think of it as the *angular rate* or *angle per unit time*.



Example: A spinning table saw blade

The circular blade of a table saw is 25 cm in diameter and spins at 3600 rpm. How much time is required for one revolution? How fast is one of the teeth at the edge of the blade moving? What is the tooth's acceleration?



Solution:

Each tooth is undergoing uniform circular motion (unless its Ryobi), so we can use the equations from the previous slide after converting minutes to seconds,

$$f = 3600\,\frac{rev}{min} \times \frac{1min}{60s} = 60\,\frac{rev}{s} = 60Hz$$

The time for one revolution is the period,

$$T = \frac{1}{f} = \frac{1}{60s^{-1}}0.017s$$

The speed of the tooth is,

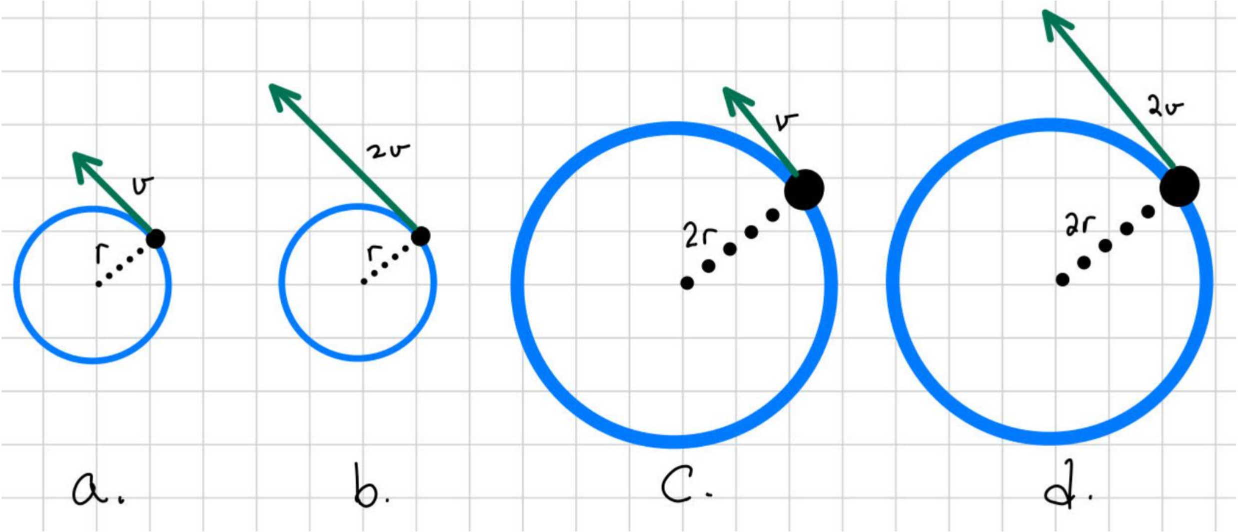
$$v = 2\pi f r = 2\pi(60s^{-1})(0.125\,m) = \boxed{47\,m/s}$$

The centripetal acceleration,

$$a = (2\pi f)^2 r = \left(2\pi(60s^{-1})\right)^2(0.125m) = \boxed{1.8 \times 10^4\,m/s^2}$$

Team Activity: Concept Check 5.3

Rank in order, from largest to smallest, the period of the motion of particles, a through d, that are undergoing uniform circular motion.



a. $v = \frac{2\pi r}{T} \rightarrow T = \frac{2\pi r}{v}$

b. $2v = \frac{2\pi r}{T} \rightarrow T = \frac{\pi r}{v}$

c. $v = \frac{2\pi(2r)}{T} \rightarrow T = \frac{4\pi r}{v}$

d. $2v = \frac{2\pi(2r)}{T} \rightarrow T = \frac{2\pi r}{v}$

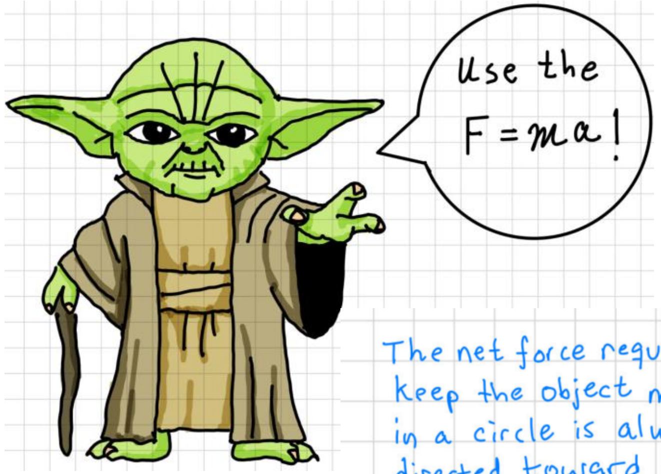
Ranking: c, a & d, b

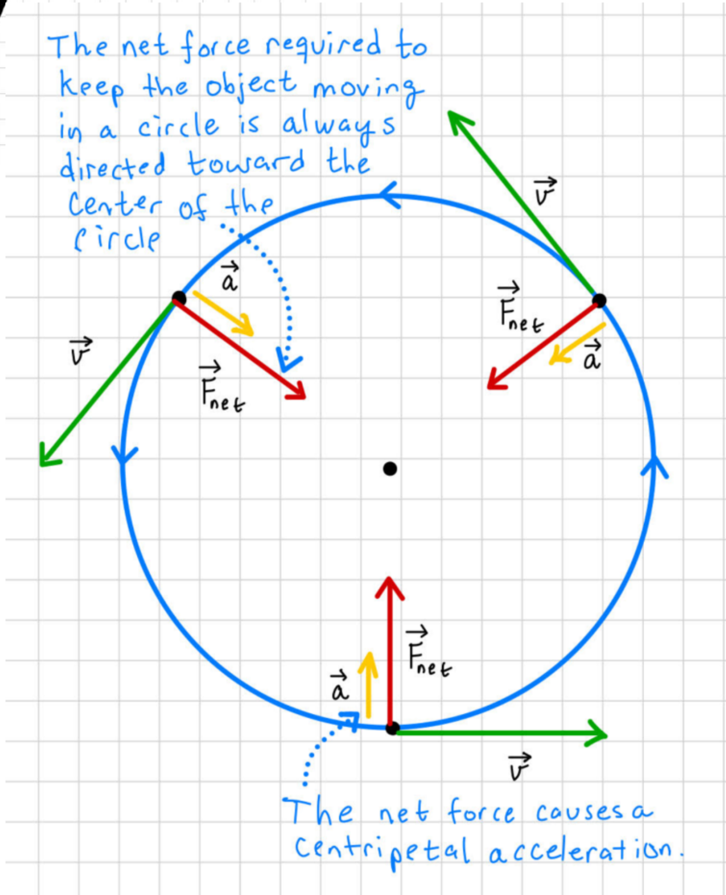
Centripetal Force

- Even in uniform circular motion, with acceleration, comes force (Newton's 2nd Law):

$$\vec{F}_{net} = m\vec{a} = \frac{mv^2}{r}\hat{r}$$

- here $\hat{r}$ is the direction toward the center of the circle.
- In other words, a particle of mass m moving at constant speed v around a circle of radius r must always have a net force of magnitude mv^2/r pointing toward the center of the circle.
- It is this net force that causes the centripetal acceleration.
  - Without it, the particle would move off in a straight line.
- The force could be tension, friction, the normal force, etc.





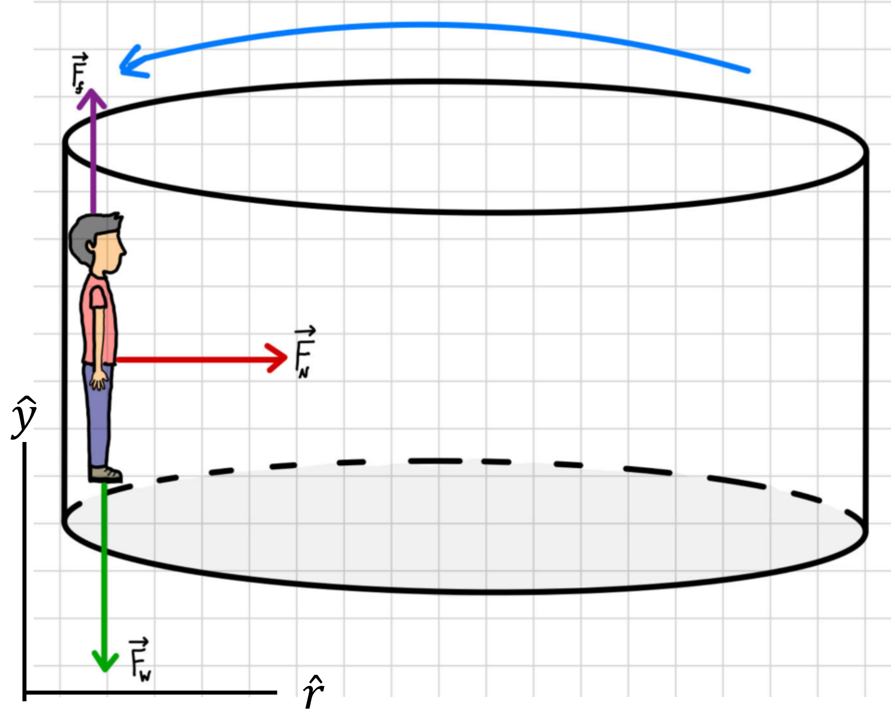
Example: Gus and The Gravitron

Gus is at the amusement park, standing in line for the Gravitron—a ride shaped like a large spinning cylinder. Riders stand with their backs against the wall as the chamber spins in uniform circular motion. Once it reaches full speed, the floor drops away, yet the riders remain "stuck" to the wall, apparently defying gravity.

Concerned about slipping, Gus wants to calculate the minimum coefficient of static friction needed to keep from sliding down the wall. He asks for your help.

The specifications of the ride are as follows:

- Radius of the chamber: $r = 8\ m$
- Frequency of rotation: $f = 24\, rev/min$
- Gus's mass: 70 kg



Solution: Sum the forces in the radial and y directions,

$$\sum F_r = F_N = \frac{mv^2}{r}$$

$$\sum F_y = F_f - F_w = ma_y \rightarrow \mu_s F_N - mg = 0 \rightarrow \mu_s \frac{mv^2}{r} = mg$$

$$\mu_s = \frac{rg}{v^2} = \frac{rg}{(2\pi f r)^2} = \frac{g}{(2\pi f)^2 r}$$

$$f = 24\frac{rev}{min} \times \frac{1min}{60s} = 0.4\frac{rev}{s} = 0.4Hz$$

$$\mu_s = \frac{g}{(2\pi f)^2 r} = \frac{9.81\ m/s^2}{\left(2\pi(0.4s^{-1})\right)^2(8\ m)} = 0.19$$

Example: Banked Curves

A car is going around a friction-free banked curve. The radius of the curve is r, where r is measured parallel to the horizontal and not to the slanted surface. The next image shows the normal force $\vec{F}_N$ that the road applies to the car, the normal force being perpendicular to the road

The force responsible for centripetal force is the force that points to the center of the circular track,

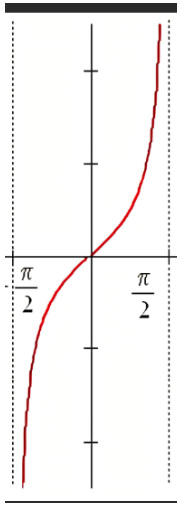
$$F_C = F_N \sin\theta = \frac{mv^2}{r}$$

The vertical component of the normal force must balance the weight of the car because there is no acceleration in the vertical direction,

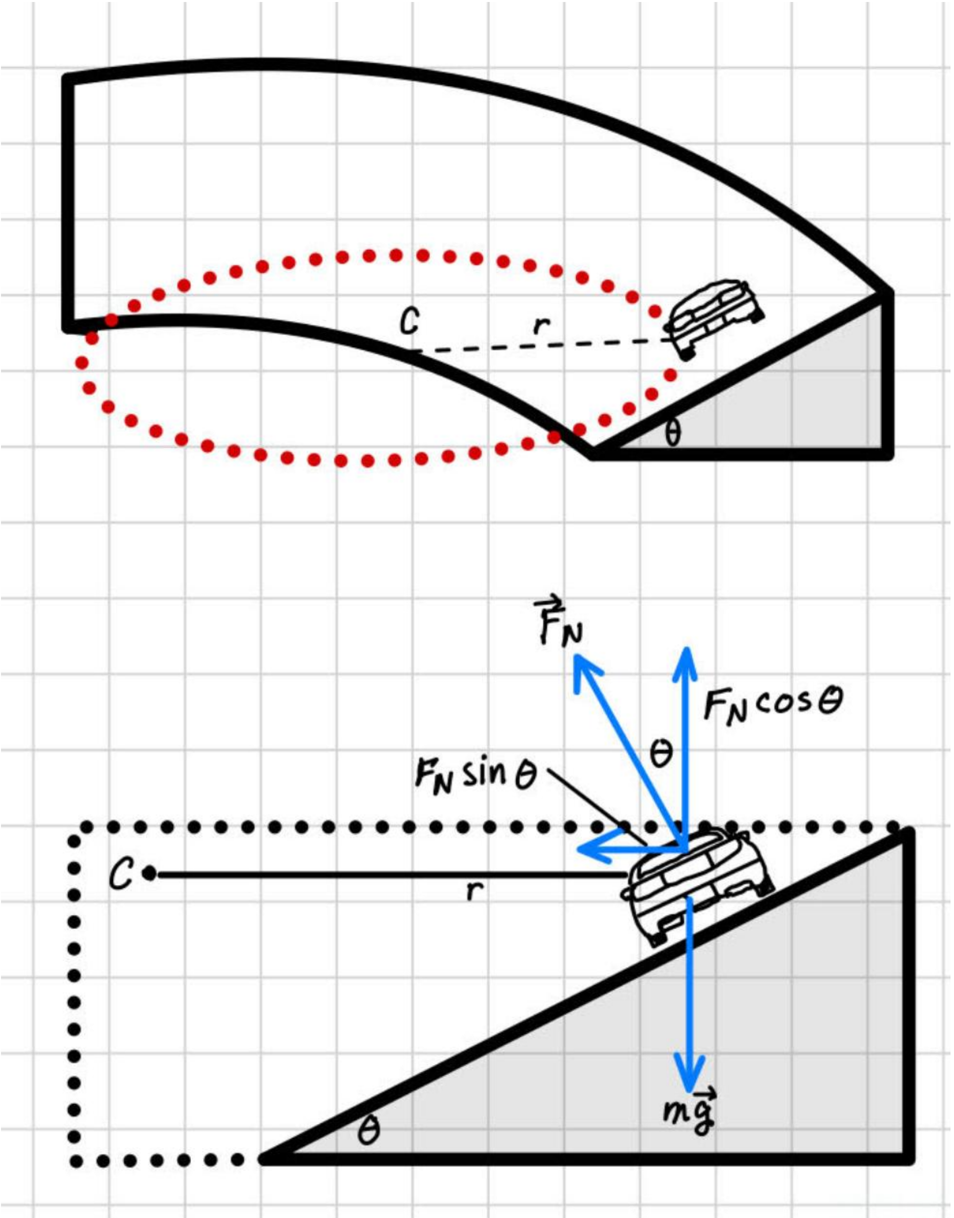
$$F_N \cos\theta = mg$$

Let's divide these two equation because why not?

$$\frac{F_N \sin\theta}{F_N \cos\theta} = \frac{mv^2/r}{mg} \rightarrow \tan\theta = \frac{v^2}{rg} \rightarrow v = \sqrt{rg\tan\theta}$$



What do greater speeds and larger values of r require to maintain centripetal motion?



Gravitational Orbits

In this problem, we will derive expressions for the orbital speed and orbital period of a satellite in circular orbit around a planet, using the principles of uniform circular motion and Newton's law of universal gravitation. To ground our derivation in a real-world example, we consider the Cassini Orbiter, a spacecraft that spent over a decade studying Saturn and its moons.

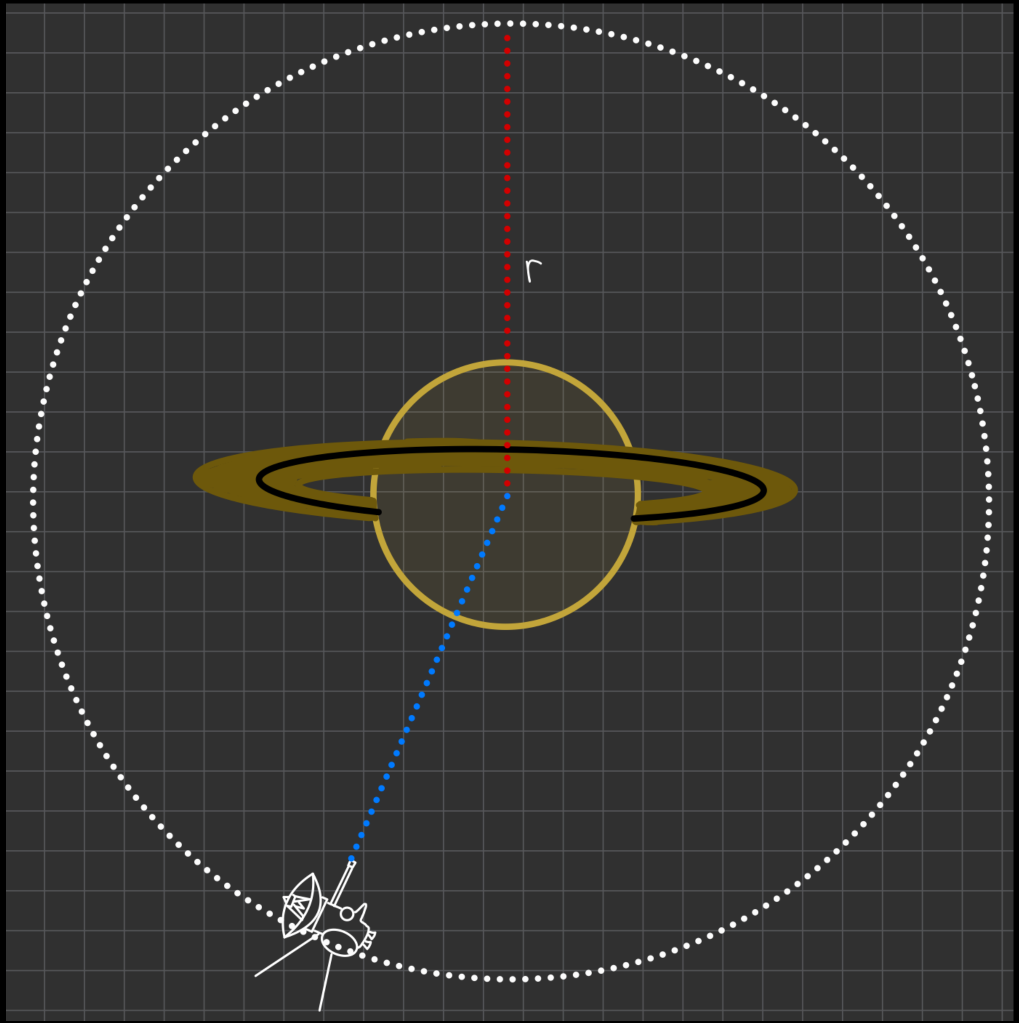
$$F_C = G\frac{mM}{r^2} = \frac{mv^2}{r} \rightarrow v = \sqrt{\frac{GM}{r}}$$

Notice the mass of the satellite does not appear – for a given orbit , a satellite with a large mass has the same orbital speed as one with small mass!

We know how period relates to velocity,

$$v = \frac{2\pi r}{T} = \sqrt{\frac{GM}{r}} \rightarrow T = \frac{2\pi r^{3/2}}{\sqrt{GM}}$$

This proportionality, $T \sim r^{3/2}$, was discovered by Johannes Kepler over 400 years ago!



Example: Locating a geostationary satellite

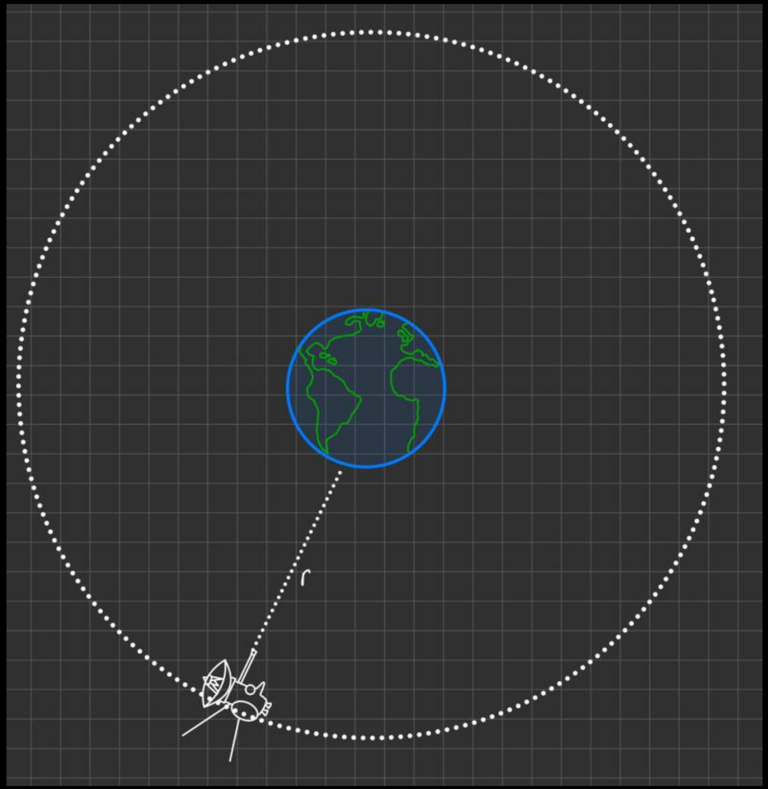
Communication satellites appear to "hover" over one point on the Earth's equator. A satellite that appears to remain stationary as the Earth rotates is said to be in a geostationary orbit. What is the radius of the orbit of such a satellite?

Solution: For a satellite to remain stationary with respect to the Earth, the satellite's period must be 24 hours.

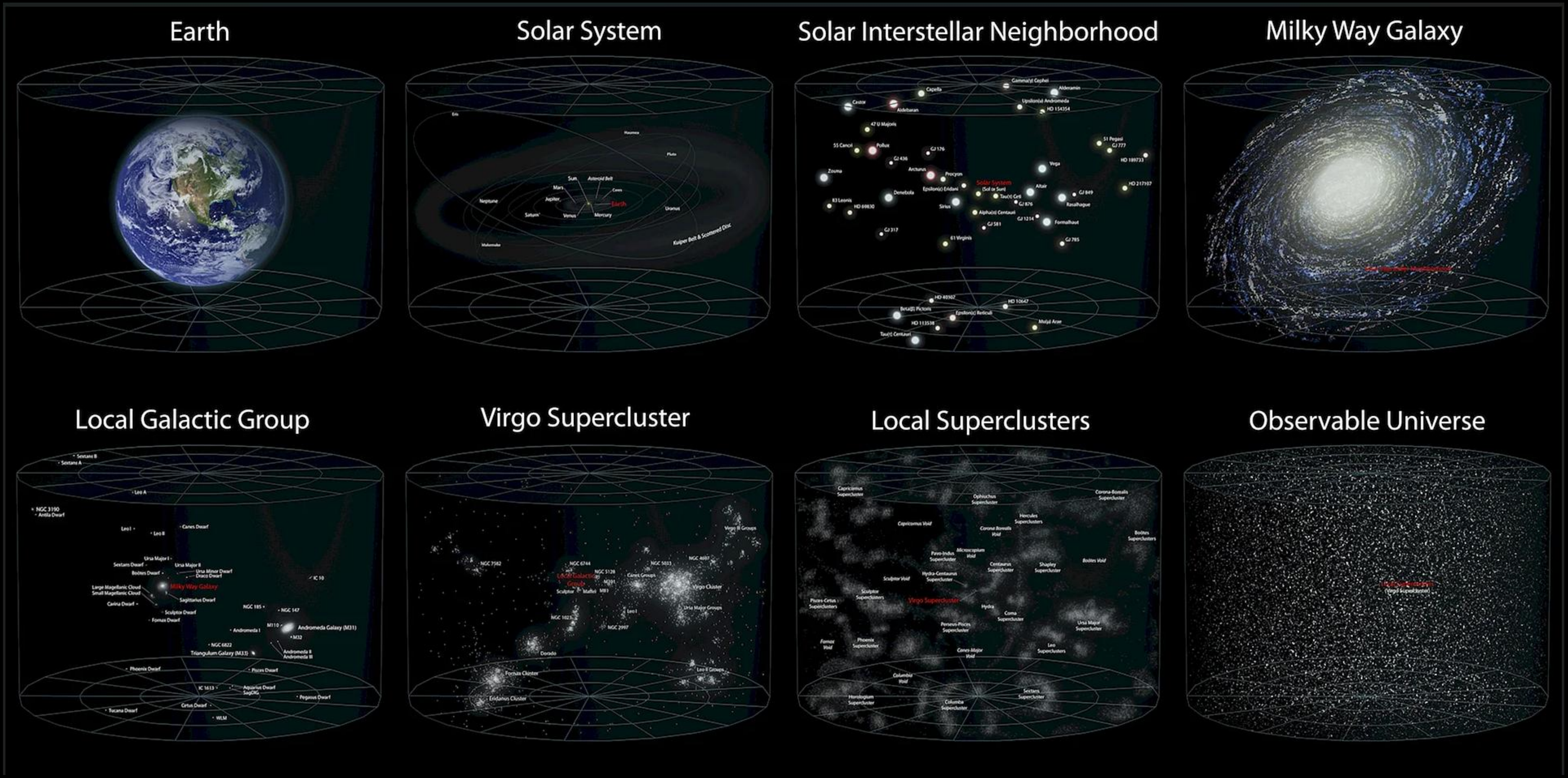
$$T = \frac{2\pi r^{3/2}}{\sqrt{GM}} \rightarrow r = \left(\frac{GM_E T^2}{4\pi^2}\right)^{1/3} = \sqrt[3]{\frac{GM_E T^2}{4\pi^2}}$$

$$r = \sqrt[3]{\frac{(6.67 \times 10^{-11}\ N \cdot m^2/kg^2)(5.98 \times 10^{24} kg)(8.64 \times 10^4 s)^2}{4\pi^2}}$$

$$= 4.23 \times 10^7 m$$



Gravity on a Grand Scale



The Hubble Ultra Deep Field

The Hubble Ultra Deep Field covers a patch of sky about the size of a grain of sand held at arm's length and contains approximately 10,000 galaxies.

Each point of light in this photo from big to small is an entire galaxy, each with billions to trillions of stars in it all interacting with each other via gravity.

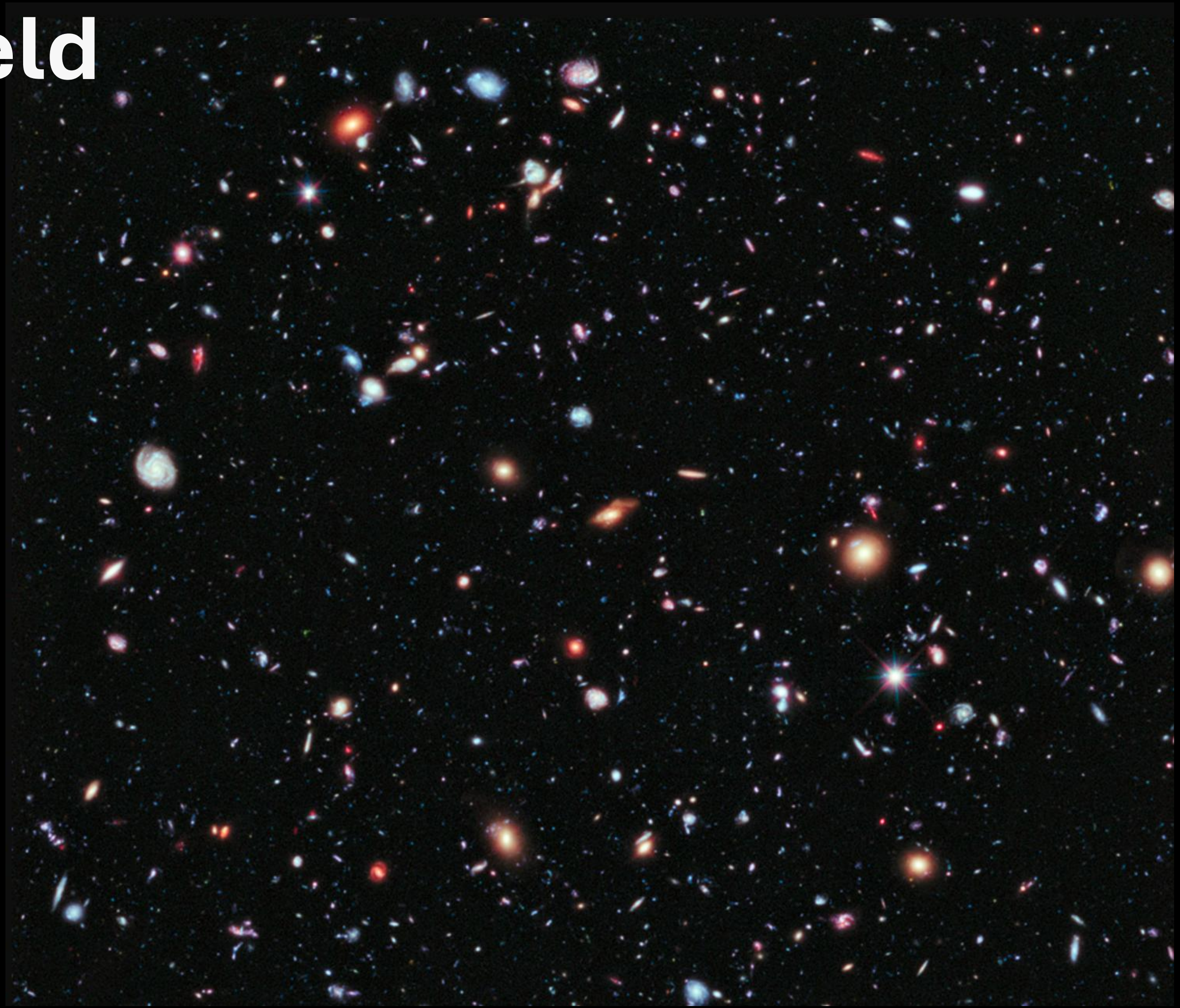
Every sand grain size of sky we point telescopes at; we see tens of thousands of galaxies.

Estimates of the number of galaxies in our observable universe:

2×10^{11} to 2×10^{12}

Stars in the observable universe: 1×10^{24}

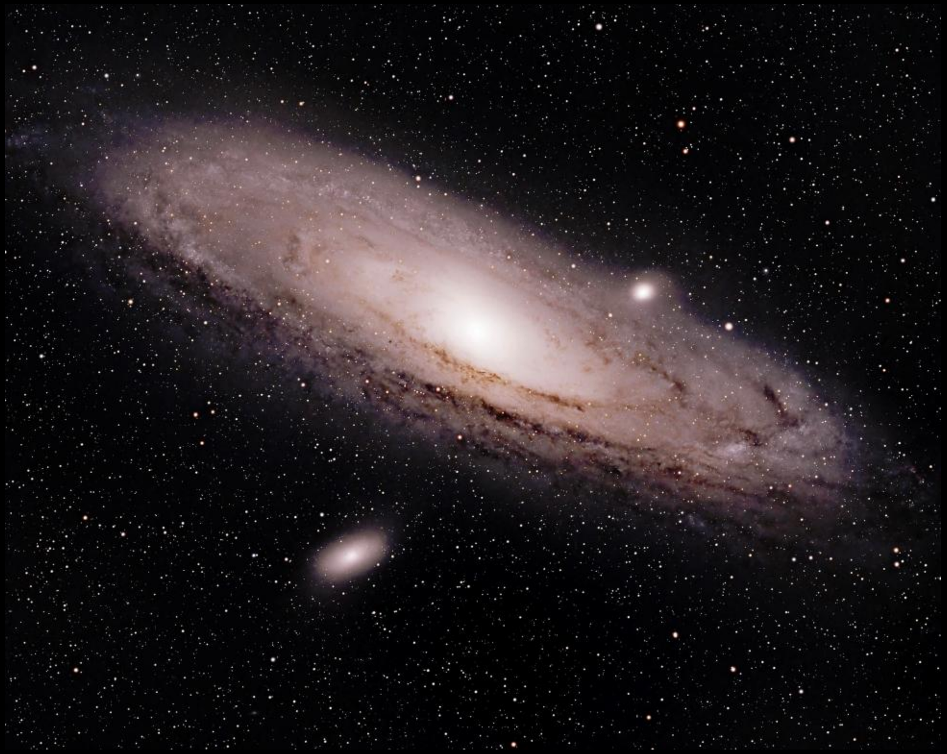
Estimated grains of sand on the Earth: 1×10^{19}



There are 100,000 times more stars in the Universe than grains of Sand on Earth!

Example: Orbital Speed of A Star in the Andromeda Galaxy

Assume that a star is orbiting the galactic center of the andromeda galaxy at a radius of $r = 30{,}000\ ly$. If the total mass of Andromeda that is contained within that radius is $M = 2 \times 10^{41}\ kg$, estimate the orbital speed of the star using Newton's laws.



Solution: Convert lightyears (ly) to meters,

$$30{,}000\ ly \times \frac{9.46 \times 10^{15} m}{1\ ly} = 2.8 \times 10^{20} m$$

$$F = ma = \frac{GMm}{r^2} = \frac{mv^2}{r} \rightarrow v = \sqrt{\frac{GM}{r}}$$

$$= \sqrt{\frac{(6.647 \times 10^{-11} N \cdot m^2/kg^2)(2 \times 10^{41} kg)}{2.8 \times 10^{20} m}} = 2.7 \times 10^5\,\frac{m}{s}$$

$$= 485{,}620\ mph$$

In our own galaxy, the Milky Way, our Sun and all of its planets, move around the galactic center with an orbital speed of

$$v = 514{,}000\ mph$$

Example: Artificial Gravity

Imagine a massive artificial ring encircling a star exactly like the Sun. The ring rotates to simulate gravity on its inner surface. What should the radius of the ring be if the artificial gravity experienced by people standing on the inner surface is equal to Earth's gravity and the ring completes one full rotation every 24 hours?

Solution: We know that the centripetal force is,

$$F_c = \frac{mv^2}{r}$$

The floor of this ring world exerts a normal force on the feet of the man in the ring world city; the normal force is the centripetal force!

$$F_N = mg = \frac{mv^2}{r} = F_C \rightarrow r = \frac{v^2}{g} = \frac{(\omega r)^2}{g} \rightarrow r = \frac{g}{\omega^2} = \frac{g}{\left(\frac{2\pi}{T}\right)^2}$$

There are $86400\ s$ in $24\ hours$,

$$r = \frac{9.81ms^2}{\left(\frac{2\pi}{86400s}\right)} = 1.85 \times 10^5\ m$$