Work and Energy

Chapter 6

The Amazing Gunjito

The approach we used to solve this works **only at the bottom of the swing** because:

- The motion is approximately circular at that point.
- Gunjito's velocity and direction of motion are known (3.0 m/s, tangential).

However, let's suppose we now want to calculate the **tension when the rope is at a 45° angle** from vertical.

Why Newton's Laws Aren't Enough (Yet)

At that point in the swing:

1. **Gunjito's speed is unknown.**
   - We only know he was moving at 3.0 m/s at the bottom.
   - To use $F = ma$, we need to know the centripetal acceleration $a_c = v^2/r$, which means we must know v at $\theta = 45^0$ and due to gravity, the speed will be different throughout.
2. **There is no simple way to relate the motion at different points in the swing using only Newton's Laws.**
   - To solve for velocity at different points, we'd need to know how the forces acted over time or distance - but our current tools don't give us a way to connect velocity at one point to another when the net force is changing constantly.

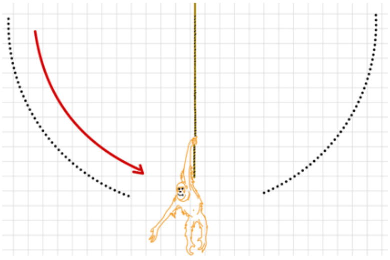
Why We Need a New Tool: What we need is a way to relate motion at one location (e.g., the bottom of the swing) to motion at another (e.g., 45°) - even if we don't know the details of how the forces changed his speed along the way.

4. The Amazing Gunjito

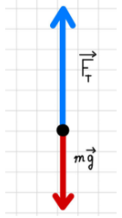
The Amazing Gunjito is an Orangutan in Borneo who frequently swings from a rope near popular tourist paths with his hand outstretched, demanding peanuts. Gunjito has a mass of 75 kg and swings from a rope that is 7m long. He passes through the lowest point at 3.0 m/s.

(a) Draw a free body diagram, assume uniform circular motion, and find the tension in the rope when Gunjito is at the lowest point.

(b) In part (a) you calculated the tension in the rope at the bottom of Gunjito's swing. Suppose we now want to calculate the tension at a different point in the swing, such as when the rope makes a 45-degree angle with the vertical. Would the same approach from part (a) still apply? Why or why not?



Solution:



(a). Sum up the forces in the y-direction at Gunjito's lowest point,

$$\sum F_y = F_T - mg = ma_c = \frac{mv^2}{r} \rightarrow F_T = mg + \frac{mv^2}{r} = m\left(g + \frac{v^2}{r}\right)$$

$$= 75\left(9.81 + \frac{9}{7}\right)N = 832.2\,N$$

(b). No, it would not work. Gunjito is not actually in uniform circular motion — only at the very bottom of the swing can we approximate it that way. At other points, like 45 degrees, his speed is lower than 3 m/s, so the expression for centripetal acceleration no longer applies without adjustment. To calculate the tension elsewhere, we'd need a different approach — one we'll explore in the next chapter on energy.

Work Done by a Constant Force

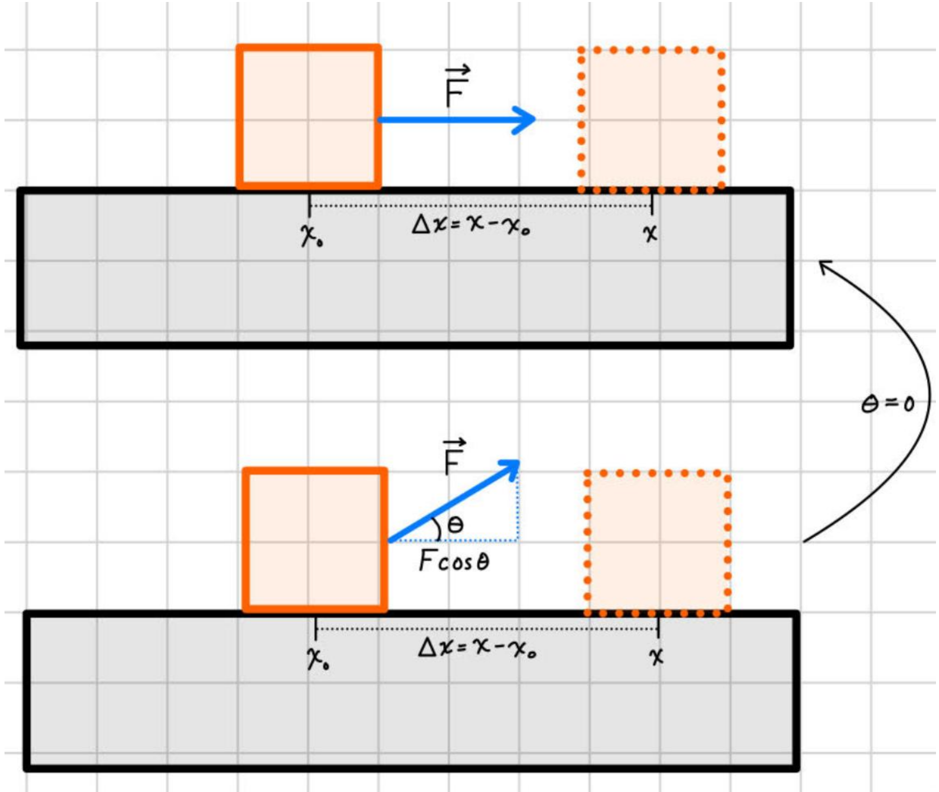
Work is how we measure the *effect* a force has when it causes something to move.

$$W = F\Delta x \cos\theta$$

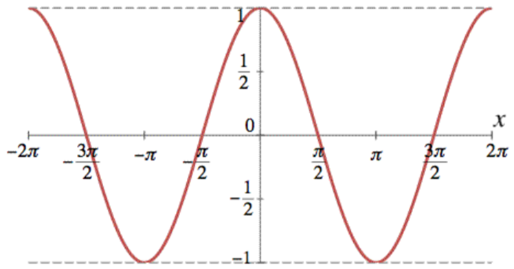
- W is work, measured in *Joules;* $1\,J = 1\,kg \cdot m^2/s^2$.
- F is force, measured in *Newtons.*
- Δx is the displacement of the object.
- θ is the angle between the force and the displacement.

Properties:

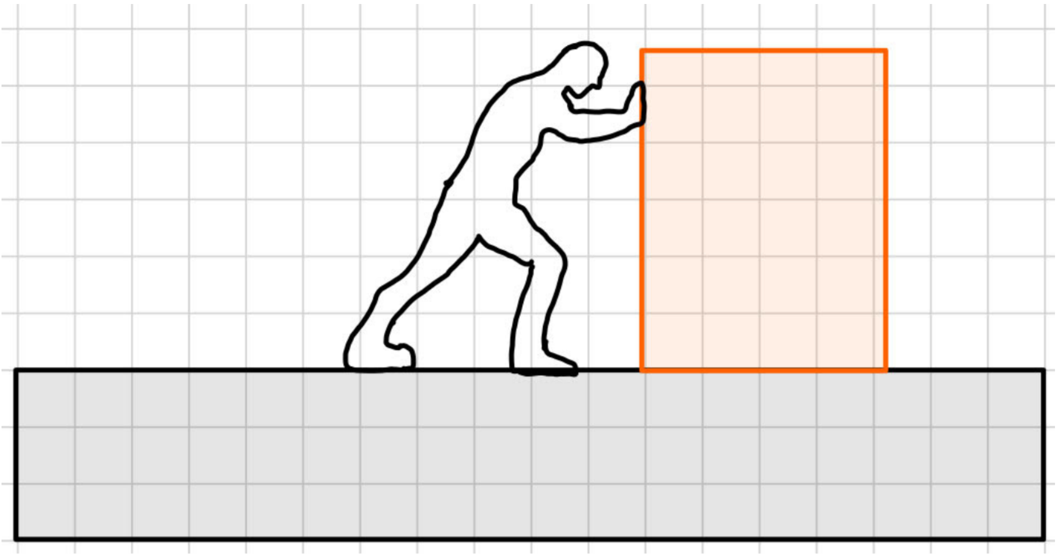
- When $\vec{F}$ and $\Delta\vec{x}$ have parallel components → positive work.
- When $\vec{F}$ and $\Delta\vec{x}$ have anti-parallel components → negative work.



Note that $\cos\theta$ *is bound between 1 and -1.*

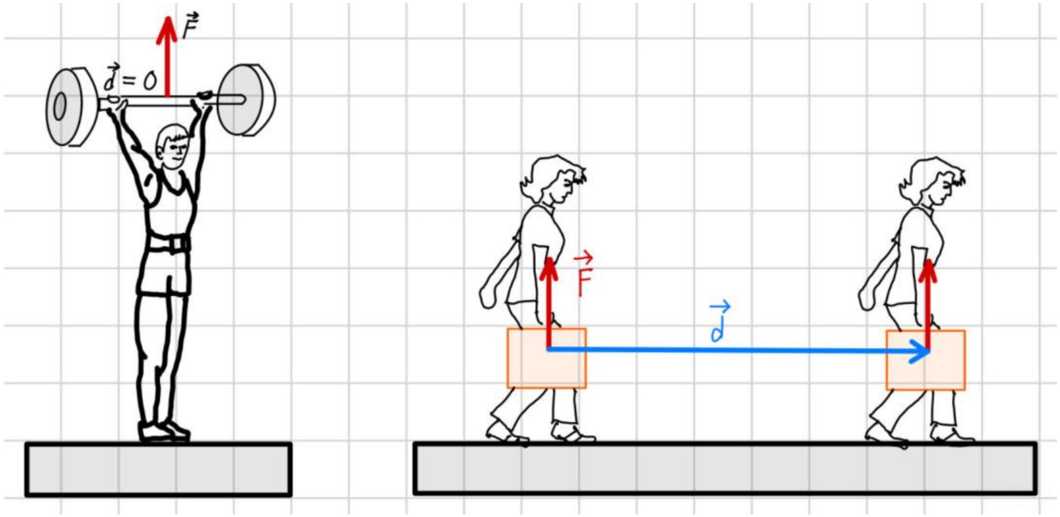


Team Activity: Concept Check 6.1



A man pushes a box across a floor. Does the resulting frictional force between the box and the floor do negative or positive work?

Team Activity: Concept Check 6.2

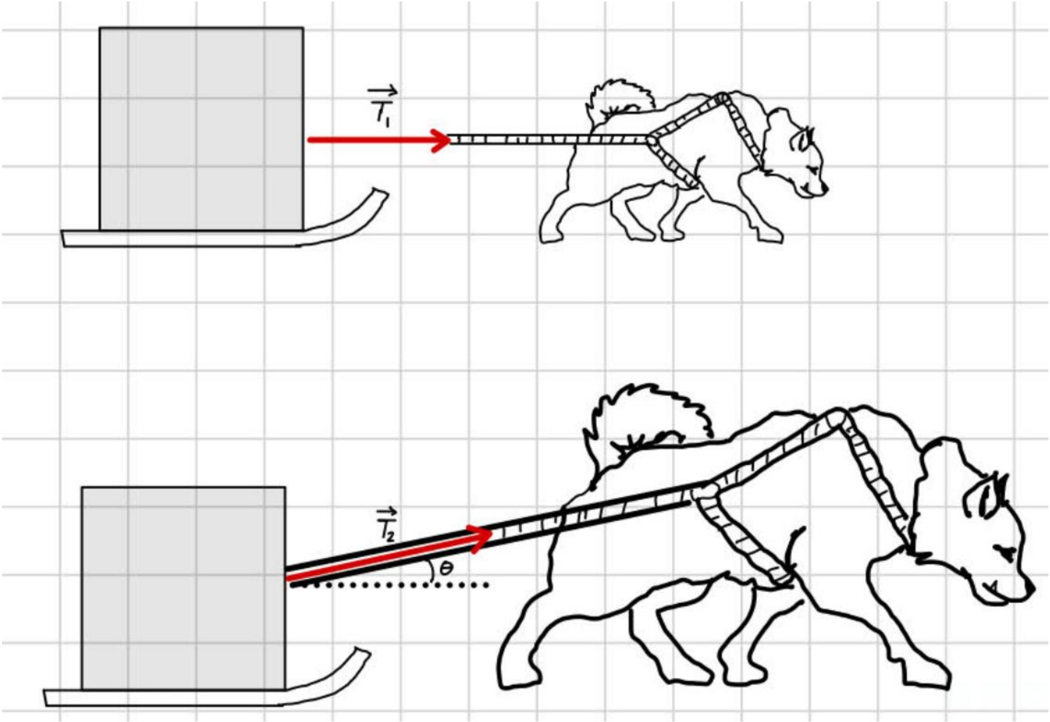


A weightlifter holds a barbell over his head. A woman carries a heavy briefcase some distance d. Who does more work:

- The weightlifter holding a barbell over his head
- The woman carrying the briefcase

Example: Work like a dog (or wolf)

A husky pulls a box on a sled a distance d with a tension of $T_1 = 200\ N$. A wolf does the same but with a tension of $T_2 = 600\ N$ at an angle θ. What is θ if the dog and wolf did the same amount of work over the distance d? Assume everything else is equal: friction, sled, box, etc.



Solution:

The dog does work: $W = T_1 d$

The wolf does work: $W = T_2 d \cos\theta$

If they do the same work: $T_1 d = T_2 d \cos\theta \rightarrow \theta = \cos^{-1}\frac{T_1}{T_2}$

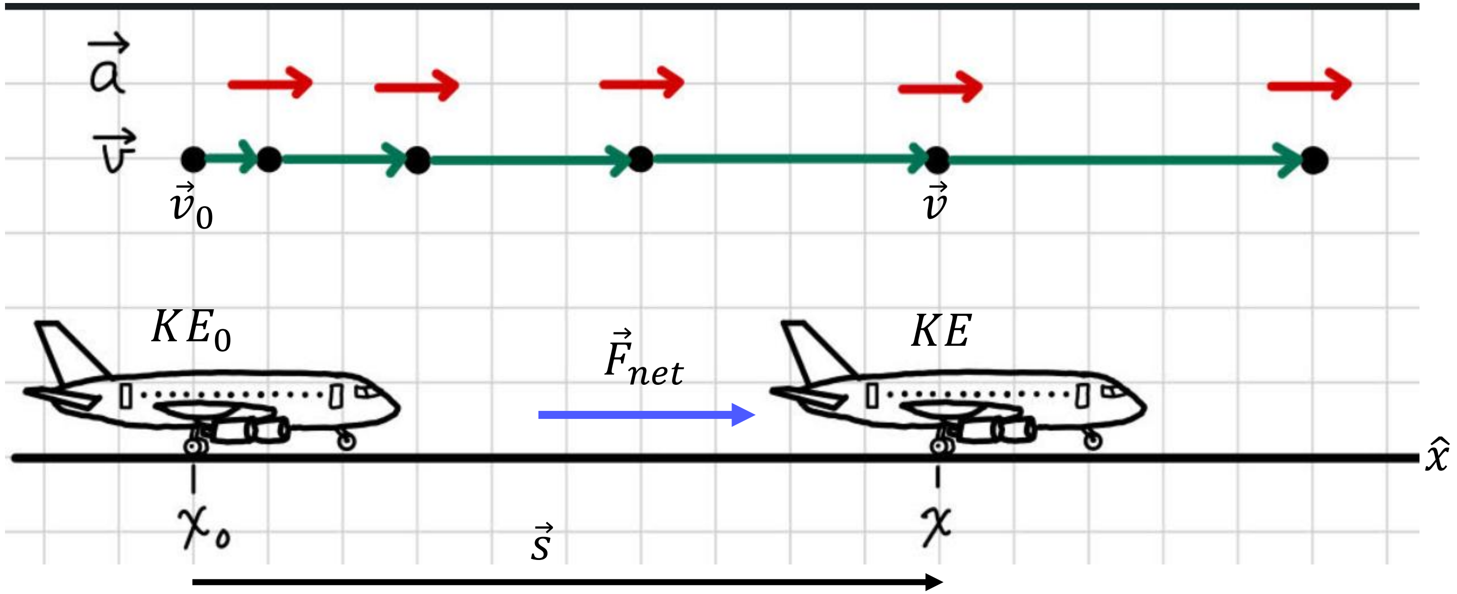
Therefore,

$$\theta = \cos^{-1}\frac{T_1}{T_2} = \cos^{-1}\frac{200}{600} = 70.5^0$$

Angle Matters!

The Work-Energy Theorem

- A net external force, $\vec{F}_{net}$, acts on an airplane of mass m moving it a distance $\vec{s} = \vec{x} - \vec{x}_0$.
  - $\vec{F}_{net} = \sum F_x \rightarrow$ sum of all external forces, for simplicity assume all in $\hat{x}$ direction.
- At x_0 it has an initial $\vec{v}_0$, at x a final $\vec{v}$.



Start with Newton's 2nd Law:

$$F_{net} = ma$$
$$W_{F_{net}} = F_{net}s = mas$$

Multiply by the displacement s to get the work done by the net external force

Recall from kinematics,

$$v^2 - v_0^2 = 2a(x - x_0) = 2as$$

Rewritten,

$$as = \frac{1}{2}(v^2 - v_0^2)$$

Now sub it into (1),

$$W_{F_{net}} = \frac{1}{2}m(v^2 - v_0^2) = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$$
$$= KE - KE_0$$

Where $KE = \frac{1}{2}mv^2$ is **KINETIC ENERGY**!

Work results from a change in kinetic energy!

Kinetic Energy and the Work-Energy Theorem

Definition of Kinetic Energy

The kinetic energy KE of an object with mass m and speed v is given by

$$KE = \frac{1}{2}mv^2$$

SI Unit of Kinetic Energy: joule (J)

The Work-Energy Theorem

When a net external force does work W on an object, the kinetic energy of the object changes from its initial value of KE_0 to a final value KE, the difference between them being equal to the work:

$$W = KE - KE_0 = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$$

Example: KE and the W-E Theorem

A 58 kg box is coasting down a 25^0 slope. Near the top of the slope its speed is 3.6 m/s. It accelerates down the slope because of the gravitational force, even though a kinetic frictional force with $\mu_k = 0.14$ opposes its motion. Ignoring air resistance, determine the speed at a point that is displaced 57 m downhill.

Solution: Sum the forces, $\sum F = mg\sin\theta - f_k$

If we multiply by the displacement we get work,

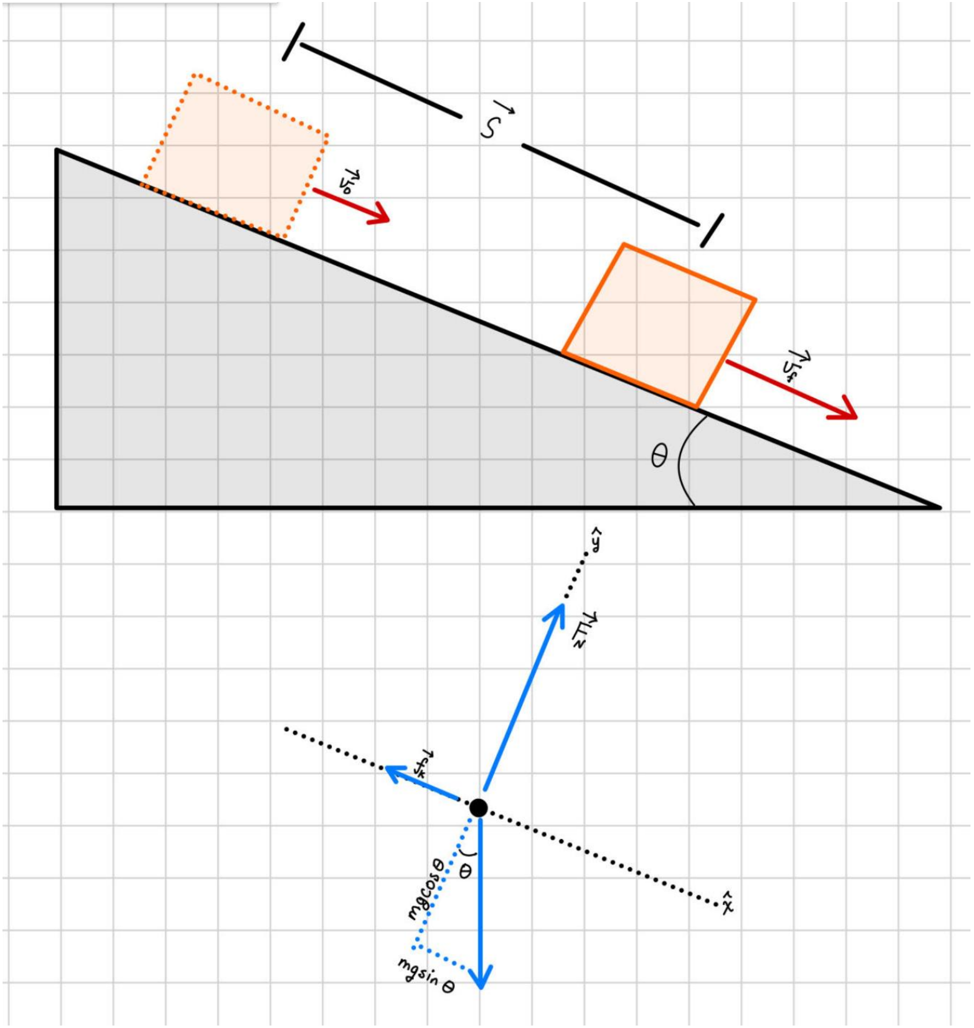
$$W = (mg\sin\theta - \mu_k mg\cos\theta)s$$

Use the Work-Energy Theorem,

$$W = KE - KE_0 = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = m(g\sin\theta - \mu_k g\cos\theta)s$$

$$v = \sqrt{v_0^2 + 2gs(\sin\theta - \mu_k\cos\theta)}$$

$$= \sqrt{(3.6)^2 + 2(9.81)(57)(\sin 25^0 - 0.14\cos 25^0)}\, m/s = 19\, m/s$$



Chapter 4 Example: Kinetic Fictional Force

The same problem but with $v_0 = 0$ because it starts from rest! Here we used Newton's 2nd Law and Kinematics, but on the previous slide we used the Work-Energy Theorem!

Consider a block with mass M sliding down and incline that is θ degrees from the horizontal surface. Derive an expression for the final velocity of the block after it travels a distance d down the incline once it is in motion.

Sum the forces in y: $\sum F_y = F_N - mg\cos\theta = 0 \rightarrow F_N = mg\cos\theta$

Sum the forces in x: $\sum F_x = ma_x = mg\sin\theta - f_k = mg\sin\theta - \mu_k F_N$

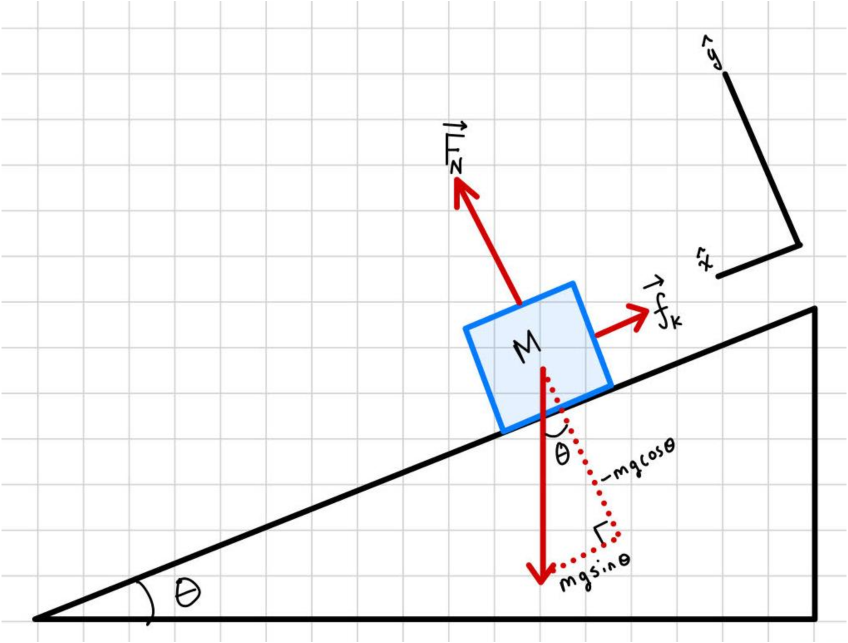
$$ma_x = mg(\sin\theta - \mu_k\cos\theta)$$

$$a_x = g(\sin\theta - \mu_k\cos\theta)$$

Now use kinematic equations,

$$v_x^2 - v_{0x}^2 = 2a_x(x - x_0) \rightarrow v_x = \sqrt{2gd(\sin\theta - \mu_k\cos\theta)}$$

This is an example of a nonequilibrium system (more on this later).



Chapter 4 Example: Kinetic Fictional Force

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Consider a block with mass M sliding down and incline that is θ degrees from the horizontal surface. Derive an expression for the final velocity of the block after it travels a distance d down the incline once it is in motion.

Sum the forces in y: $\sum F_y = F_N - mg\cos\theta = 0 \rightarrow F_N = m$...

Sum the forces in x: $\sum F_x = m$...

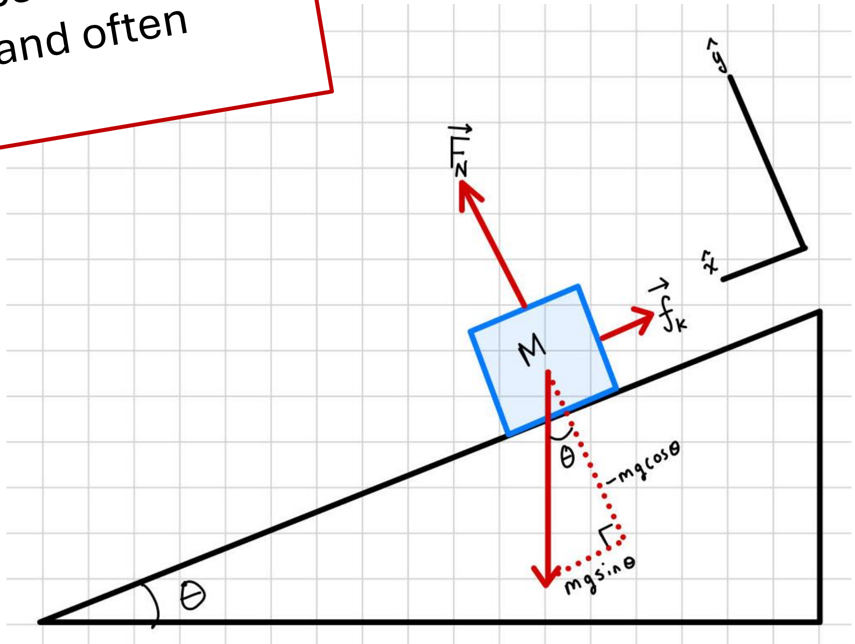
The Work-Energy Theorem is a **compressed version** of Newton's 2nd Law **plus** kinematic relationships. It's derived from them, but it is more **flexible** and often more **efficient.**

m...

$$a_x = g(\sin\theta - \mu_k\cos\theta)$$

Now use kinematic equations,

$$v_x^2 - v_{0x}^2 = 2a_x(x - x_0) \rightarrow v_x = \sqrt{2gd(\sin\theta - \mu_k\cos\theta)}$$



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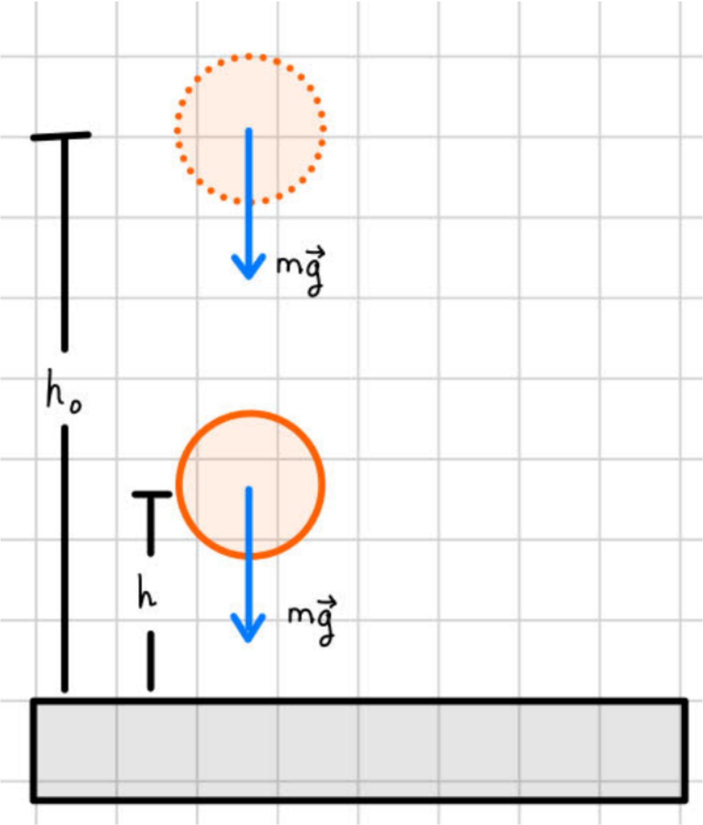
Gravitational Potential Energy

Consider a basketball dropped from some height h_0. It falls due to gravity, a force. Define $\hat{y}$ to be up, then the distance the ball falls is, $h - h_0$. Therefore,

$$W_{gravity} = F_{gravity}s = -mg\cos 0^0\,(h - h_0) = mgh_0 - mgh$$



Notice that we have a change in some quantity, and we get work! But there is no velocity involved this time – so its not kinetic energy.

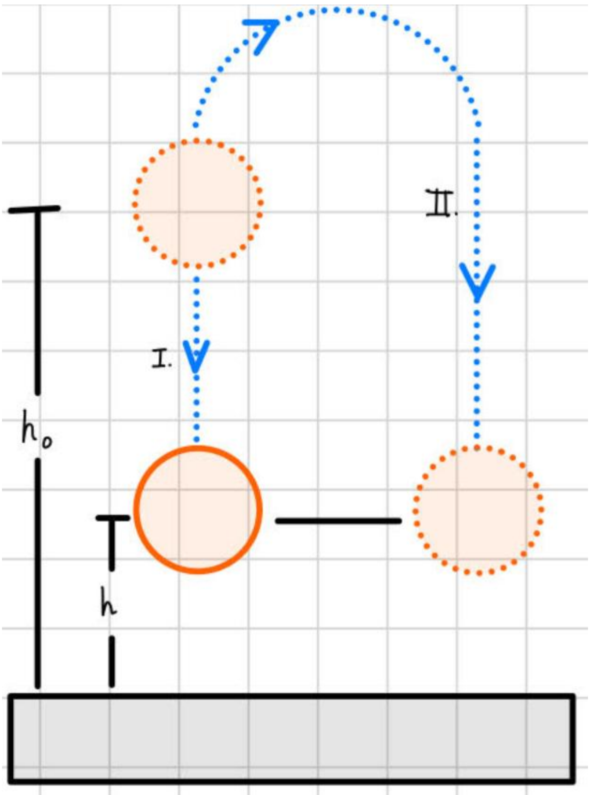
Gravitational Potential Energy

The gravitational potential energy PE is the energy that an object of mass m has by virtue of its position relative to the surface of the earth. That position is measured by the height h of the object relative to an arbitrary zero level:

$$PE = mgh$$

SI Unit of Gravitational Potential Energy: Joule (J)

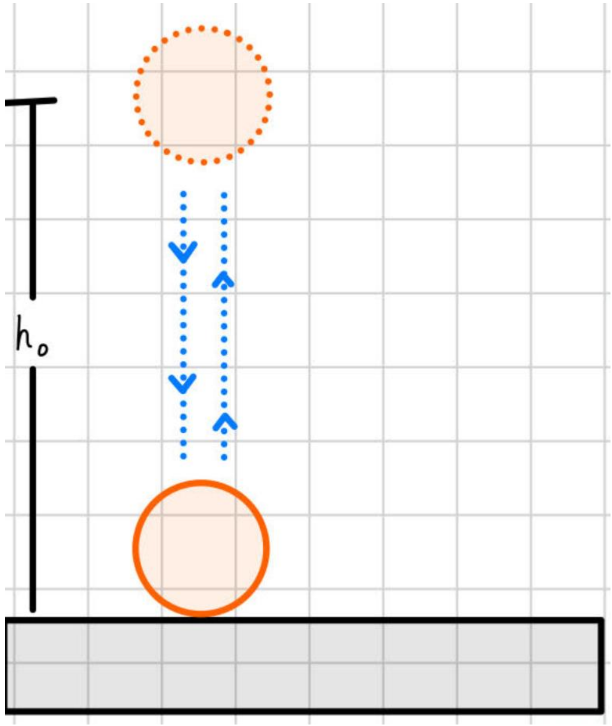
Team Activity: Concept Check 6.3



There are two paths shown in the diagram that the basketball could take from the initial height to the final height. Path I drops straight down, its motion completely in the $\hat{y}$ direction, while path II goes above h_0 with velocity components in both directions and then eventually down to h. Which path requires more work done by gravity?

$$W_{gravity} = mgh_0 - mgh$$

Team Activity: Concept Check 6.4

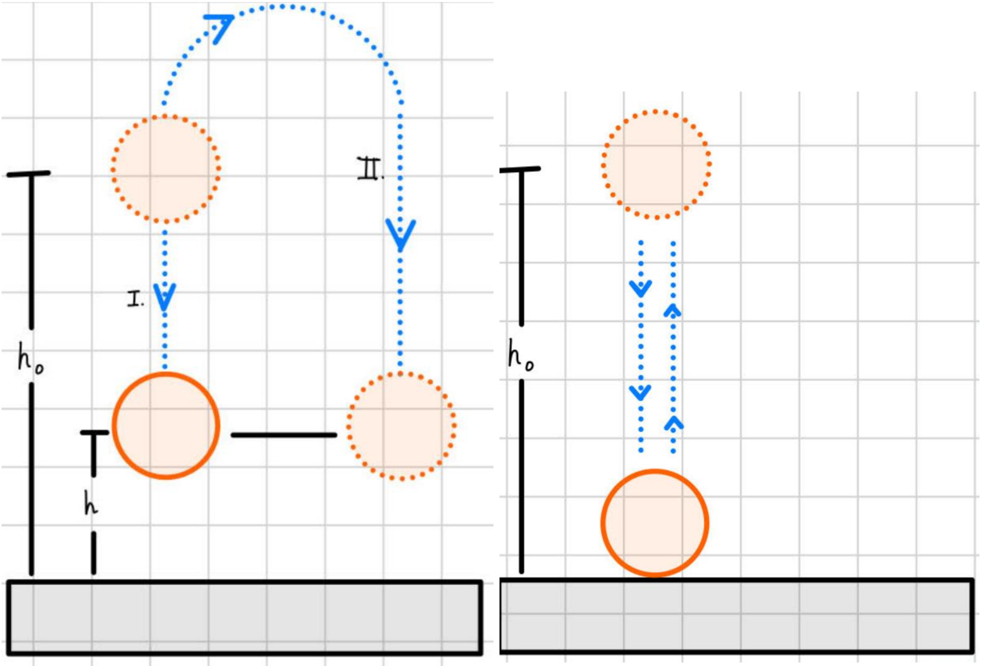


What is the work done by gravity on the basketball if it drops from a height h_0 bounces off the court floor, and in the absence of air resistance, returns to its initial height h_0?

$$W_{gravity} = mgh_0 - mgh$$

Assume that there is no deformation in the basketball when it collides with the floor.

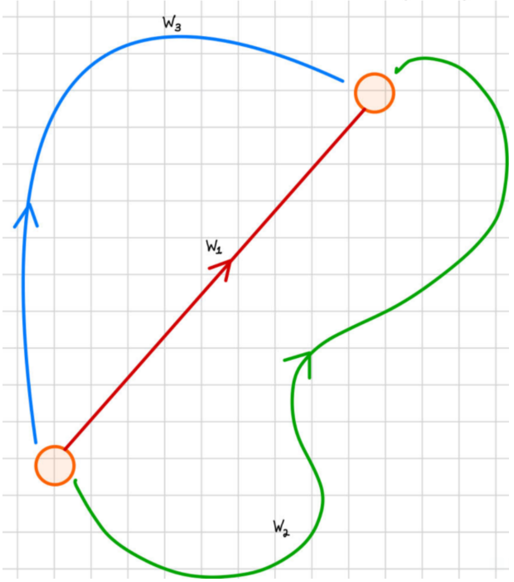
Conservative Forces



Definition of a Conservative Force

Version 1: A force is conservative when the work it does on a moving object is independent of the path between the object's initial and final positions.

Version 2: A force is conservative when it does no net work on an object moving around a closed path, starting and finishing at the same point.



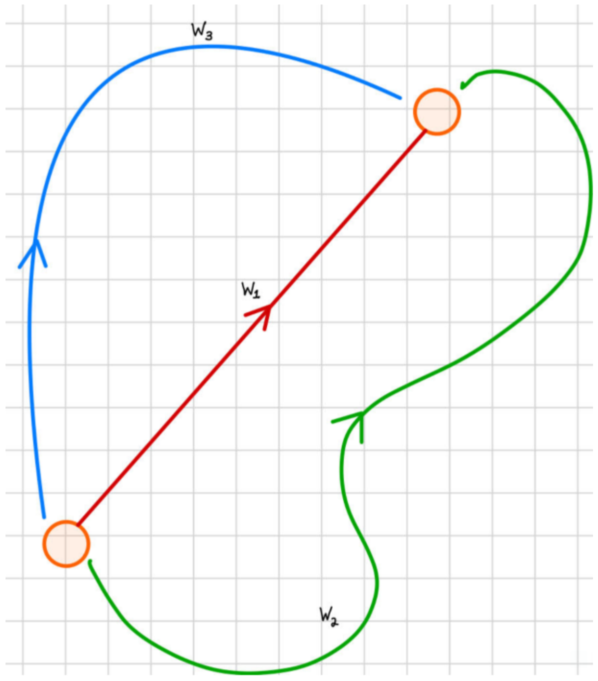
Examples of conservative forces:

- Gravitational force
- Elastic spring force
- Electrical force

3 paths between two points: Path 1 (red), Path 2 (green), Path 3 (blue).

$$W_1 = W_2 = W_3$$

Nonconservative Forces



Definition of Nonconservative Forces

Version 1: A force is nonconservative if the work it does on an object moving between two points depends on the path of the motion between the points.

Version 2: A force is nonconservative when it does non-zero net work on an object moving around a closed path, starting and finishing at the same point.

The concept of potential energy is undefined for a nonconservative force.

Examples:

- Frictional force
- Air resistance
- Tension
- Normal force
- Propulsion force

Friction:

- When you push a block across a surface with friction, your applied force does positive work, adding energy to the system.
- However, friction does negative work, converting some of that mechanical energy into heat—and that energy is permanently lost from the system.
- If you take a longer or rougher path, friction removes more energy, because the work it does depends on the path, not just the start and end points.

3 paths between two points: Path 1 (red), Path 2 (green), Path 3 (blue).

$$W_1 \neq W_2 \neq W_3$$

Work Due to Conservative and Nonconservative Forces

Consider a system with conservative and nonconservative forces contributing to the motion of some object.

$$W = W_C + W_{NC}$$

According to the work energy theorem, $W = \Delta KE$,

$$W = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = W_C + W_{NC}$$

If the only conservative force acting on the object is the gravitational force, $W_C = mg(h_0 - h)$,

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = mg(h_0 - h) + W_{NC} \rightarrow W_{NC} = \left(\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2\right) + (mgh - mgh_0)$$

$$\boxed{W_{NC} = \Delta KE + \Delta PE}$$

Generalized Work-Energy Theorem

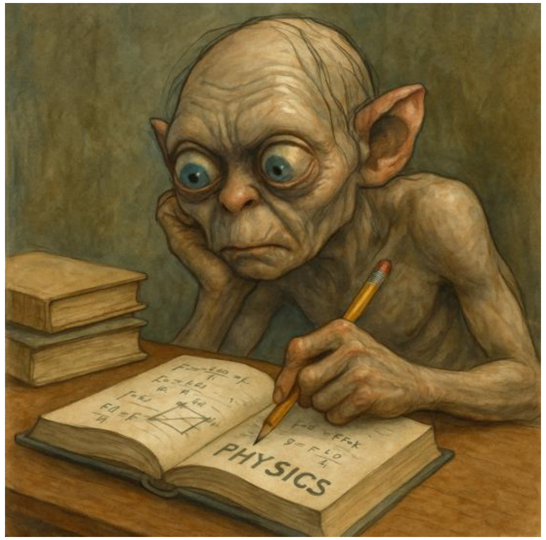
The net work W_{NC} done by all the external nonconservative forces equals the change in the object's kinetic energy plus the change in its gravitational potential energy.

The Conservation of Mechanical Energy

From the previous slide,

$$W_{NC} = \left(\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2\right) + (mgh - mgh_0)$$

$$= \left(\frac{1}{2}mv^2 + mgh\right) - \left(\frac{1}{2}mv_0^2 + mgh_0\right) = E_f - E_0$$



Energy cannot be created or destroyed, but it can change forms!

Suppose that $W_{NC} = 0\,J$, meaning that the net work by all nonconservative forces is zero (either because there are none or they cancel each other out),

$$W_{NC} = 0 = E_f - E_0 \rightarrow E_f = E_0$$

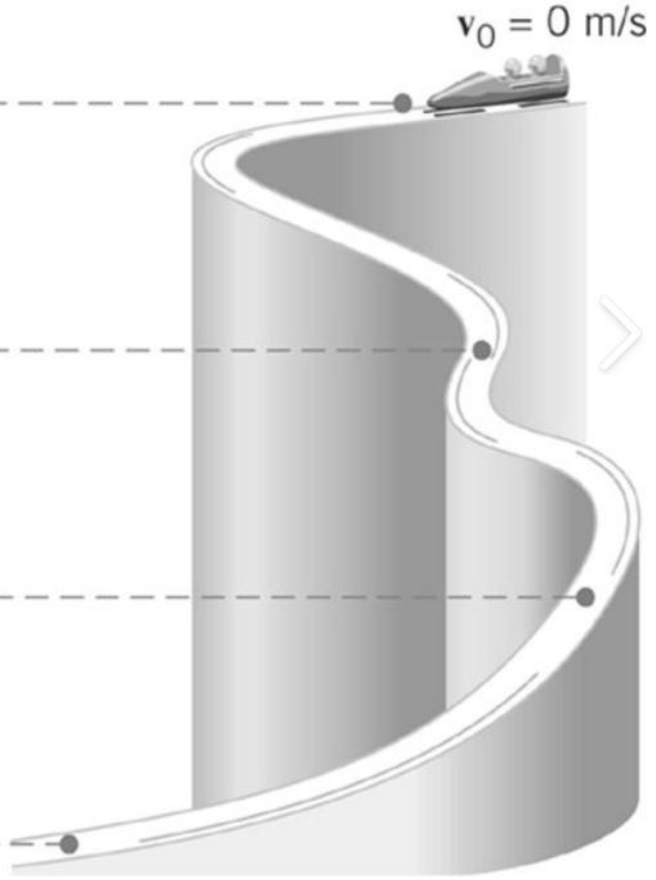
$$\frac{1}{2}mv^2 + mgh = \frac{1}{2}mv_0^2 + mgh_0$$

The Principle of Conservation of Mechanical Energy

The total mechanical energy ($E = KE + PE$) of an object remains constant as the object moves, provided that the net work done by external nonconservative forces, $W_{NC} = 0\,J$.

If friction and wind resistance are ignored, a bobsled run illustrates how kinetic energy can be converted to potential energy, while the total mechanical energy remains constant.

KE	PE	E = KE + PE
0 J	600 000 J	600 000 J
200 000 J	400 000 J	600 000 J
400 000 J	200 000 J	600 000 J
600 000 J	0 J	600 000 J



Example: The Conservation of Mechanical Energy

A block is sliding down a frictionless ramp. At position x_0 it is at a height h_0 with velocity v_0. At position x it is at a height h with a velocity v. Derive an expression for the final velocity v.

Method 1: *Sum the forces, 2nd Law* $\sum F_x = mg\sin\theta = ma \rightarrow a = g\sin\theta$

Use kinematics: $v^2 - v_0^2 = 2a(x - x_0) = 2sg\sin\theta \rightarrow v = \sqrt{v_0^2 + 2sgsin\,\theta}$

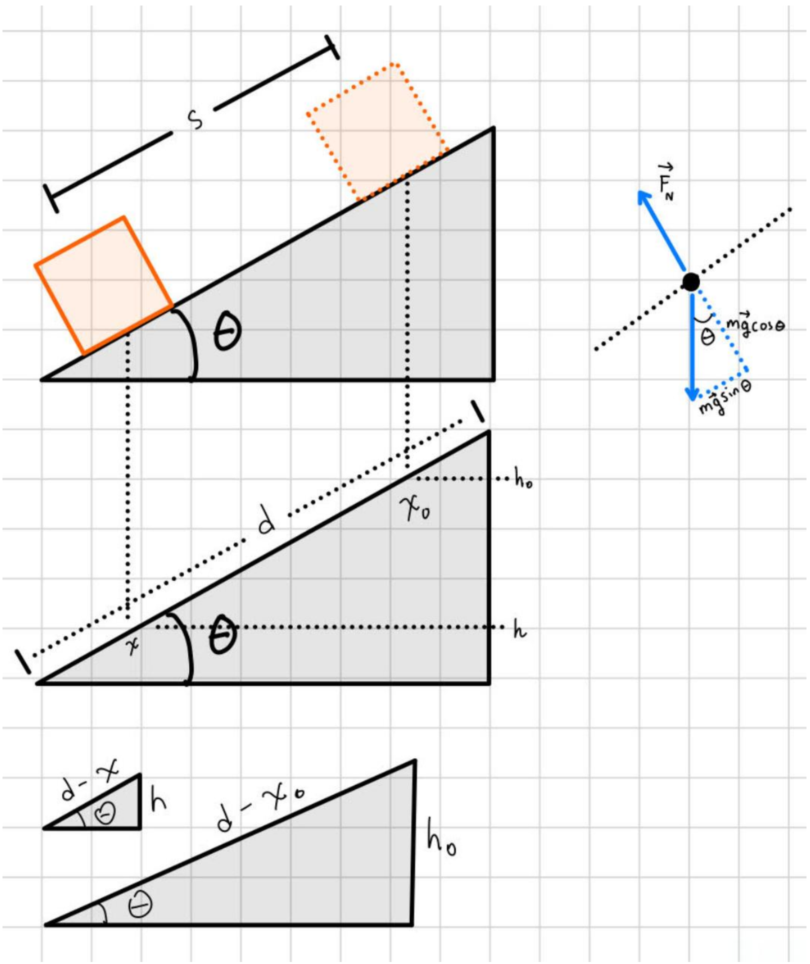
Method 2: Use the Work-Energy Theorem,

$$\frac{1}{2}mv^2 + mgh = \frac{1}{2}mv_0^2 + mgh_0 \rightarrow v^2 = v_0^2 + 2g(h_0 - h)$$

We can write h and h_0,

$$h_0 = (d - x_0)\sin\theta \qquad h = (d - x)\sin\theta \ \rightarrow h_0 - h = (x - x_0)\sin\theta$$

$$v^2 = v_0^2 + 2g(x - x_0)\sin\theta \rightarrow v = \sqrt{v_0^2 + 2sgsin\,\theta}$$



The Amazing Gunjito Revisited

We can use the conservation of energy to derive and expression for the velocity at any angle in Gunjito's swing.

First, lets assume he starts from 90^0 and swings down to the lowest part,

$$KE_0 + PE_0 = KE_F + PE_F$$

$$0 + mgl = \frac{1}{2}mv^2 + mgl(1 - \cos\theta) \rightarrow v = \sqrt{2gl\cos\theta}$$

Now we can finally get the tension in the rope at any point in Gunjito's swing,

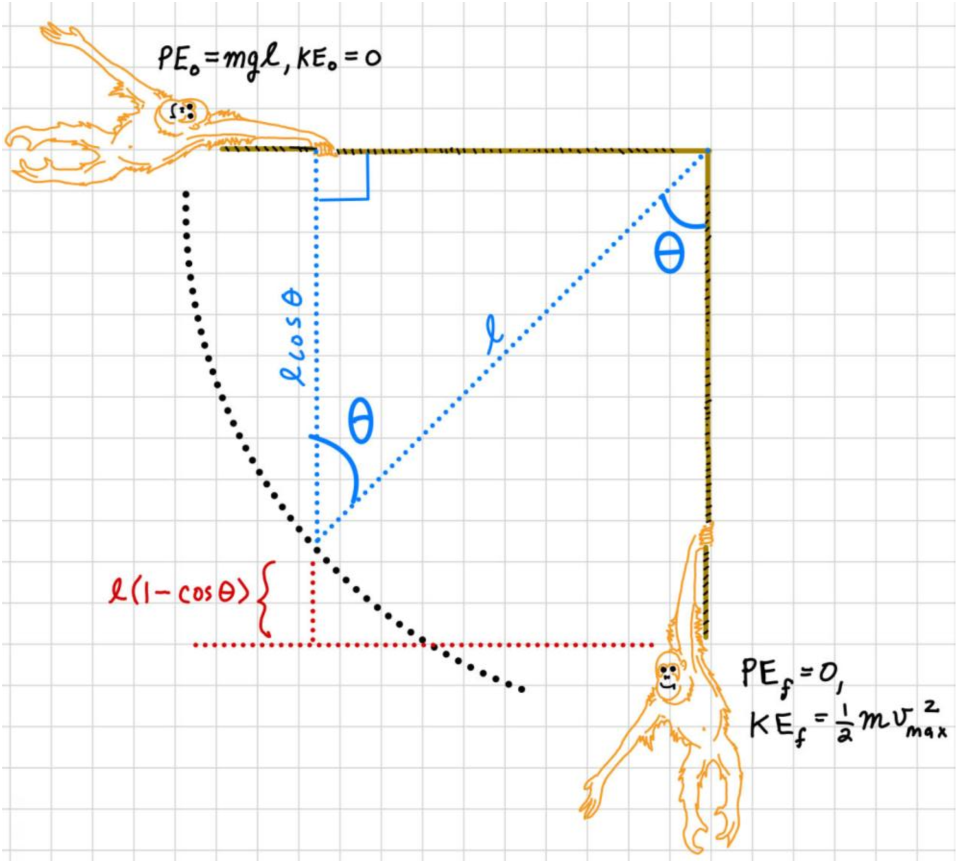
$$\sum F_r = F_T - mg\cos\theta = ma_c = \frac{mv^2}{l} = \frac{2}{l}gml\cos\theta = 2gm\cos\theta$$

$$F_T = 3gm\cos\theta$$

Notice the maximum velocity occurs at $\theta = 0 \rightarrow v_{max} = \sqrt{2gl\cos 0^0} = \sqrt{2gl}$

Notice the minimum velocity occurs at $\theta = 90^0 \rightarrow v_{min} = \sqrt{2gl\cos 90^0} = 0$

Gunjito is going to get those peanuts!



To solve this with just Newton's 2nd Law would require a much more complicated approach involving calculus and solving a differential equation – this is much easier!

Nonconservative Forces and Work-Energy Theorem

Let's revisit the block on the ramp but this time lets permit friction between the block and the ramp surface. We can no longer set $W_{NC} = 0\,J$ because friction is a nonconservative force. We begin with,

$$W_{NC} = \left(\frac{1}{2}mv^2 + mgh\right) - \left(\frac{1}{2}mv_0^2 + mgh_0\right) = E_f - E_0$$

Just as before we write h in terms of θ,

$$h_0 - h = (x - x_0)\sin\theta = s\sin\theta$$

Solving the W-E Theorem, for v,

$$v = \sqrt{\frac{2}{m}W_{NC} + v_0^2 + 2sg\sin\theta}$$

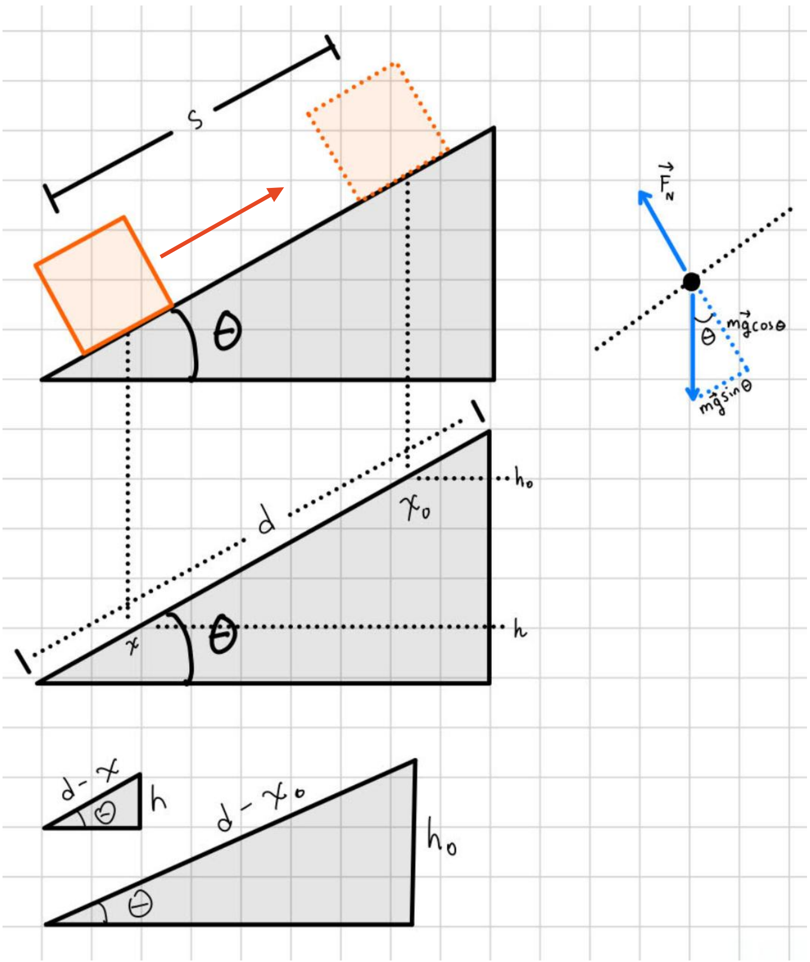
Notice this is identical to the example from the previous slide when $W_{NC} = 0\,J$. In this case we know the form of W_{NS} due to the 2nd Law and our study of friction,

$$W_{NC} = F_{net,c} \cdot s = -\mu_k F_N \cdot s = -\mu_k mg\cos\theta \cdot s$$

Plugging that into our expression for velocity,

$$v = \sqrt{v_0^2 + 2gs(\sin\theta - \mu_k\cos\theta)}$$

This is the exact result we got back on slide 9 showing consistency. You won't always be able to calculate W_{NC} but sometimes it can be measured and used in problems.



Example: Nonconservative forces in free fall

Consider a basketball with a mass of 0.6 kg in free fall in Earth's atmosphere. It is dropped at a height of 1010 m and has a measured speed of 79 m/s at 10 m. Using the W-E theorem with $W_{NC} = 0$ to compare a calculated v to our measured $v_{measured}$.

$$\frac{1}{2}mv^2 + mgh = \frac{1}{2}\cancel{mv_0^2} + mgh_0 \rightarrow v = \sqrt{g(h_0 - h)} = \sqrt{9.81(1010 - 1000)} = 99\, m/s$$

Uh oh! Our v doesn't match what was given in the problem – in fact, that is a huge difference. That is because there is air resistance. Now we don't yet have a model for air resistance yet, but we can use our measurement of v estimate the contribution of W_{NC},

$$W_{NC} = \left(\frac{1}{2}mv_{measured}^2 + mgh\right) - \left(\frac{1}{2}\cancel{mv_0^2} + mgh_0\right) = m\left(g(h - h_0) + \frac{1}{2}v_{measured}^2\right) = 0.6\left(9.81(-1000) + \frac{1}{2}(79)^2\right)$$
$$= -4014\, J$$

Now we know that -4014 J of energy was lost to air resistance. As expected, the negative sign means that nergy was removed from the system by the environment.

> *Note: If you lived in a world without vacuums, air tracks, or frictionless carts, wouldn't you think objects naturally slow down and stop?*

Power

- Consider two cars with the same mass.
  - Car 1 does 0-60 mph in 4 seconds
  - Car 2 does 0-60 mph in 8 seconds
- Each car does the same amount of work, W, but car 1 does it more quickly!
- Quicker performance in cars is associated with horsepower:
  - A large horsepower means an engine can do a larger amount of work in a short time.
  - just one way to describe something called **power.**

Definition of Average Power

Average power $\bar{P}$ is the average rate at which work W is done, and it is obtained by dividing W by the time t required to perform the work:

$$\bar{P} = \frac{W}{t}$$

SI Unit of Power: Joule / s = watt (W)

Think of average power as the change in energy per unit time.

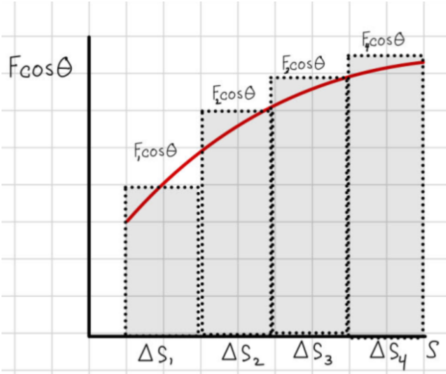
1 horsepower = 550 foot-pounds/second = 745.7 watts

Work Done by a Variable Force

- In this chapter we only considered work done by a constant force (constant acceleration).
- Many forces are variable:
  - Spring forces (a function of distance)
  - Gravity (a function of distance – can be approximated near Earth as a constant)
  - Air resistance (a function of velocity)
  - Magnetic force (Physics 203)
  - Buoyant force (Physics 202)
  - Tension in a swinging pendulum (a function of angle – more in Physics 202)
- The "quick and dirty" way to approximate the work:

$$W \approx \bar{F} \cos\theta \cdot \Delta s$$

Graphically, if we plot $F \cos\theta$ vs Δs for a constant force, the area under the line equals the total work, $W = lh = F\cos\theta \cdot \Delta s$

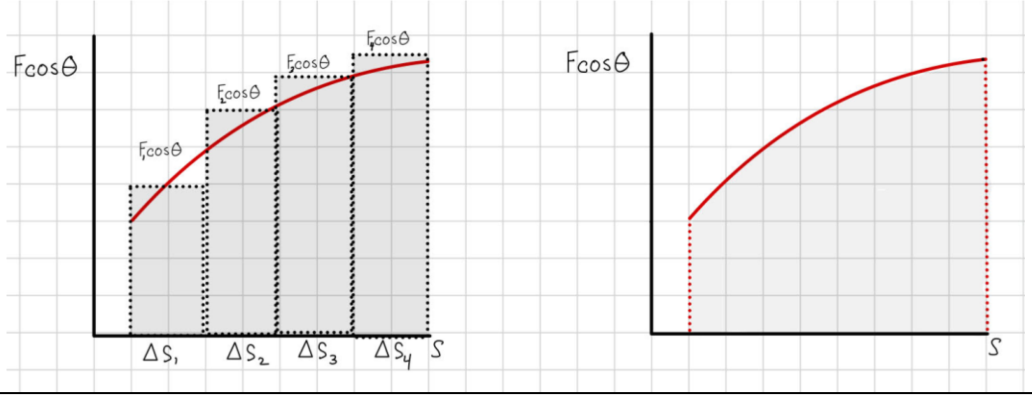


However, if F is variable, there is no simple formula in algebra to find the area. However, we could imagine that the area under the curve is made up of rectangles that roughly approximate it

$$W \approx F_1 \cos\theta \cdot \Delta s_1 + F_2 \cos\theta \cdot \Delta s_2 + \cdots = \sum_{i=1}^{N} F_i \cos\theta \, \Delta s_i$$

Team Activity: Concept Check 6.5

Using the graphical method, what are two things that you can do to increase the accuracy of approximating work?



$$W \approx F_1 \cos\theta \cdot \Delta s_1 + F_2 \cos\theta \cdot \Delta s_2 + \cdots = \sum_{i=1}^{N} F_i \cos\theta \, \Delta s_i$$

In calculus this is the idea of the integral,

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} F(x_i) \Delta x_i \cos\theta = \int_a^b \vec{F}(x) \cdot d\vec{x}$$

Energy: Final Thoughts

Is energy real or just math?

In Newtonian physics,

- Energy is a derived quantity
- It measures the capacity to do work based on forces over distances.

In deeper physics, ***energy becomes foundational***.

- Forces arise from changes in energy.
- There is no need for Newton's laws.
- In fact, in our most modern understanding of gravity, there are no gravitational forces – just geometry!

Force is nature's response to energy being uneven. *If energy is balanced, nothing moves. If energy is "lopsided," things accelerate to even it out. Systems tend toward lower energy states!*