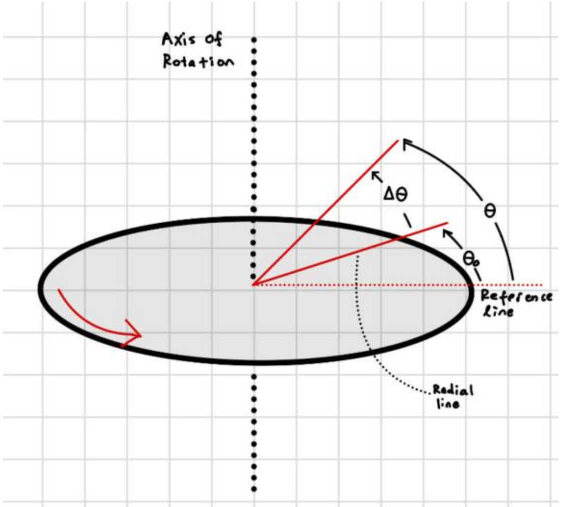
Rotational Kinematics

Chapter 8

Rotational Motion and Angular Displacement



Definition of Angular Displacement

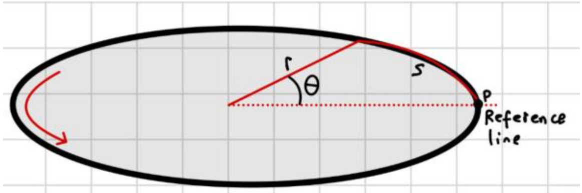
When a rigid body rotates about a fixed axis, the angular displacement is the angle $\Delta\theta$ swept out by a line passing through any point on the body and intersecting the axis of rotation perpendicularly. By convention, ***the angular displacement is positive if it counterclockwise and negative if it is clockwise.***

Consider a thin rotating disc

- **Axis of rotation:** A line that passes through the center of the disc and is perpendicular to its surface.
- **Radial lines:** Lines along the surface, from the center to the edge of the disc and perpendicular to the axis of rotation
- **Reference line:** A horizontal line from the center along the surface that does not move with the disc.

A radial line, initially aligned along the reference line moves to a new position as the disc rotates, sweeping out the angle θ_0 from the reference line, and continues to rotate with the disc until it sweeps out the angle θ from the reference line. The angular position between these two positions,

$$\Delta\theta = \theta - \theta_0$$



The arc length, s, and the radius r, are related to the angle θ,

$$\theta(in\ radians) = \frac{s}{r}$$

To convert from degrees to radians:

$$C = 2\pi r$$

For one full revolution,

$$\theta = \frac{C}{r} = \frac{2\pi r}{r} = 2\pi\ rad$$

$$1\ rad = \frac{360^0}{2\pi}$$

Team Activity: Example Challenge 8.1

Synchronous or “stationary” communications satellites are put into an orbit whose radius is $r = 4.23 \times 10^7\ m$. The orbit is in the plane of the equator, and two adjacent satellites have and angular separation of $\theta = 2.00^0$. Find the arc length s that separates the satellites.

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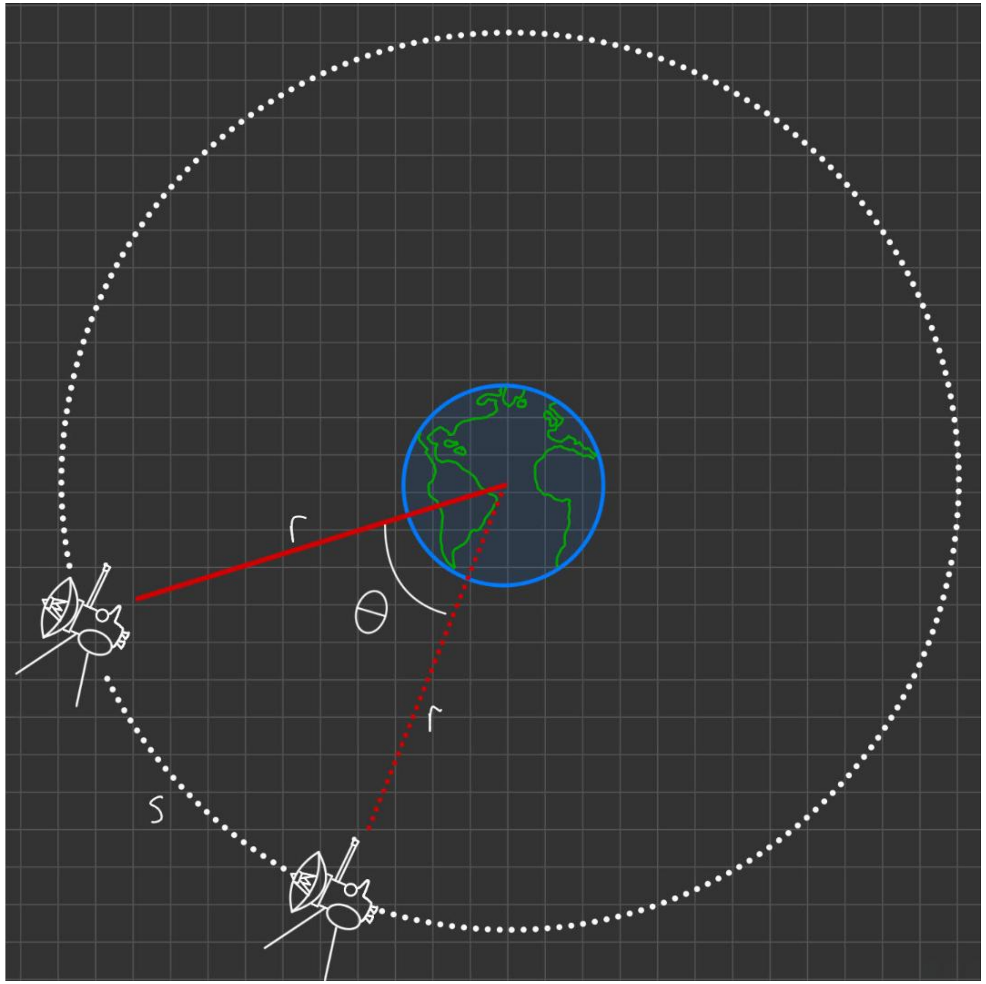


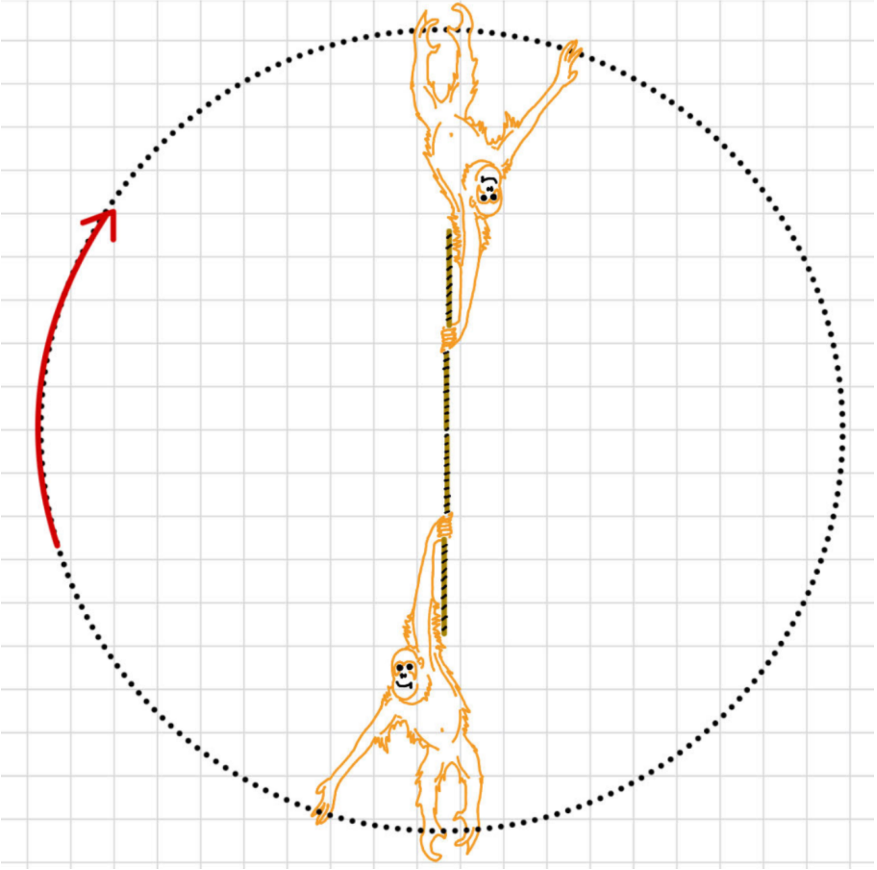
Average Angular Velocity

Definition of Average Angular Velocity

$$\bar{\omega} = \frac{\theta - \theta_0}{t - t_0} = \frac{\Delta\theta}{\Delta t}$$

SI Unit of Angular Velocity: radian per second (rad/s)

The angular displacement is positive if it counterclockwise and negative if it is clockwise.



Example 8.1: The Amazing Gunjito swings through two revolutions in a time of 1.90 s. Find the average velocity (in rad/s) of Gunjito.

Solution:

$$\Delta\theta = -2.00\left(\frac{2\pi\, rad}{1\, rev}\right) = -12.6\, rad$$

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t} = \frac{-12.6\, rad}{1.90\, s} = -6.63\, rad/s$$

Instantaneous angular velocity:

$$\omega = \lim_{\Delta t \to 0} \bar{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

Angular Acceleration

Definition of Average Angular Acceleration

$$\bar{\alpha} = \frac{\omega - \omega_0}{t - t_0} = \frac{\Delta\omega}{\Delta t}$$

SI Unit of Average Angular Acceleration: rad/s^2

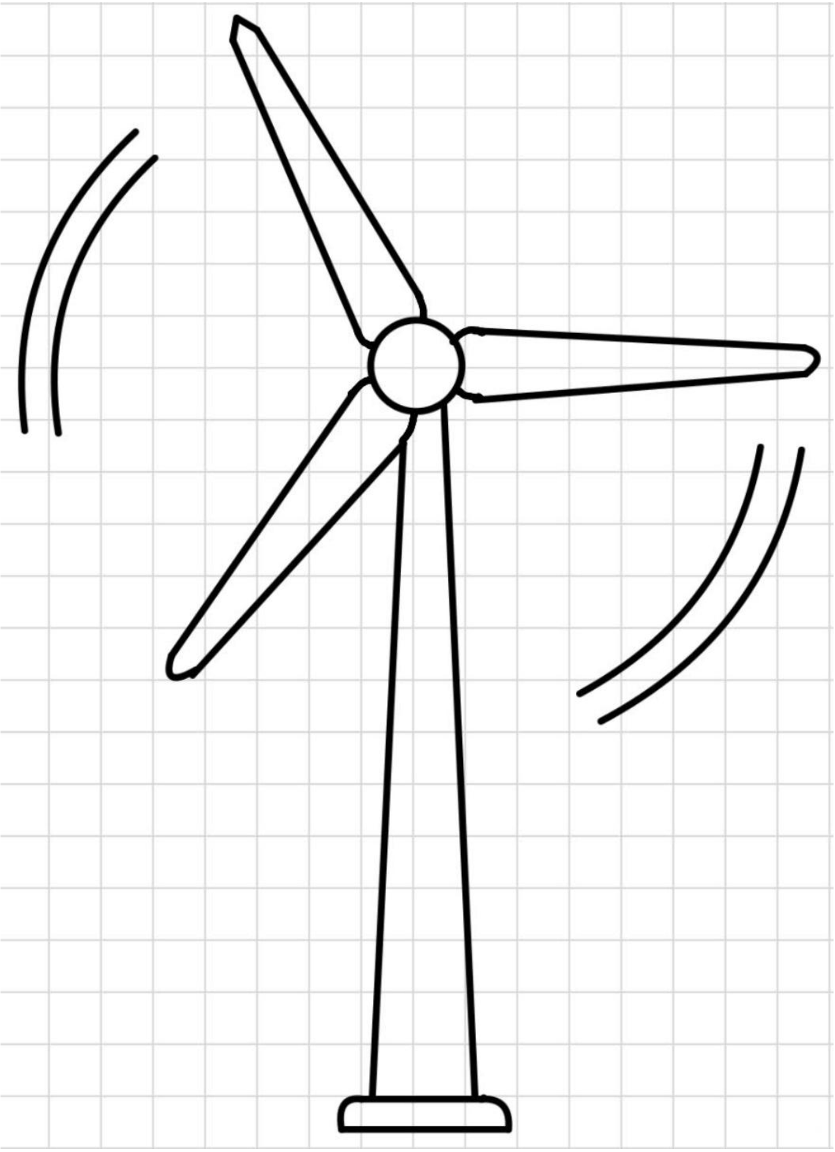
Example 8.2: A wind turbine starts from a stationary state and accelerates up to an angular velocity of $-1.6\ rad/s$ in 2 seconds. What is the average angular acceleration of the blades in revolutions per minute squared?

Solution:

$$\bar{\alpha} = \frac{\omega - \omega_0}{t - t_0} = \frac{-1.6 - 0}{2\ - 0} = -0.8\ rad/s^2$$

$$-0.8\frac{rad}{s^2} \times \frac{1}{2\pi} \times \frac{(60s)^2}{m^2} = -458.4\ rev/m^2$$

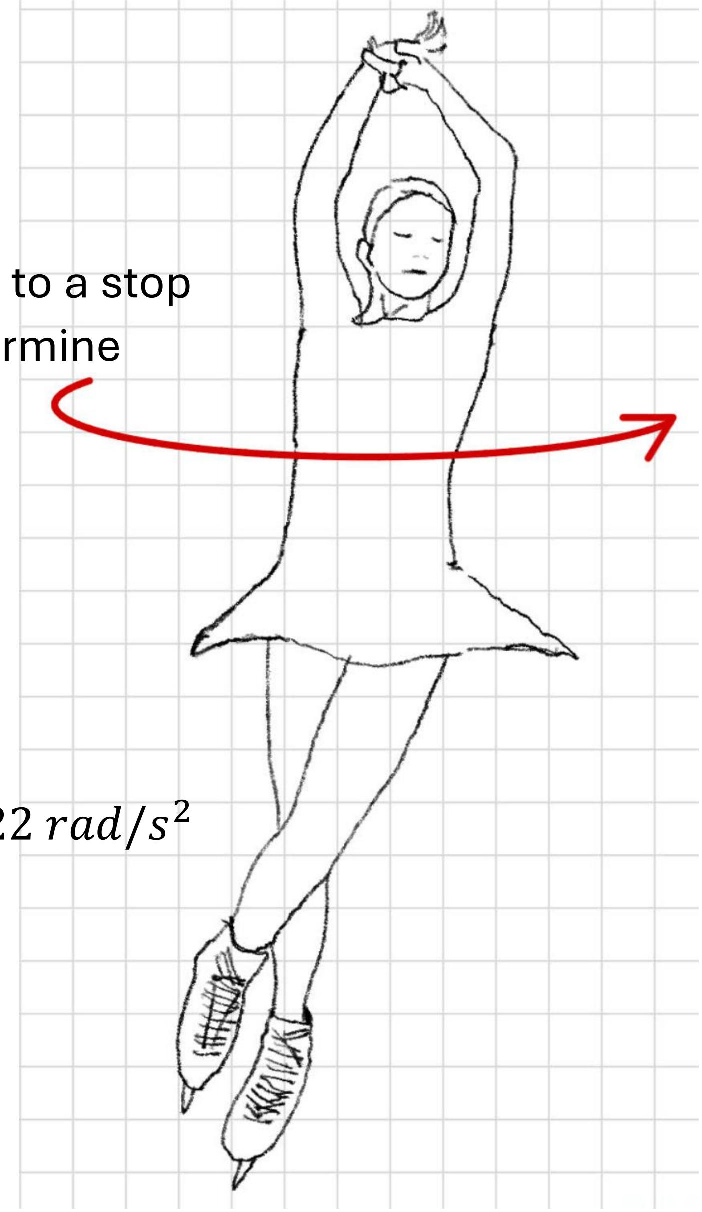
What does the negative sign mean?



The Equations of Rotational Kinematics

Linear Motion $(a = constant)$	**Rotational Motion** $(\alpha = constant)$
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = \frac{1}{2}(v_0 + v)t$	$\theta = \frac{1}{2}(\omega_0 + \omega)t$
$x = v_0 t + \frac{1}{2}at^2$	$\theta = \omega_o t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2ax$	$\omega^2 = \omega_0^2 + 2\alpha\theta$

Example 8.3: The Figure Skater



A figure skater is spinning with an angular velocity of 15 rad/s. She then comes to a stop over a brief period. During this time, her angular displacement is 5.1 rad. Determine

a. Her average acceleration, and,

b. The time during which she comes to rest.

Solution: Use the rotational kinematic equations!

Knowns: $\theta_0 = \omega = 0$ and $\omega_0 = 15\, rad/s, \theta = 5.1\, rad$

a. $\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \rightarrow 0 = \omega_0^2 + 2\alpha\theta \rightarrow \alpha = -\frac{\omega_0^2}{2\theta} = -\frac{(15)^2}{2(5.1)} rad/s^2 = -22\, rad/s^2$

b. $\theta = \frac{1}{2}(\omega_0 + \omega)t \rightarrow \theta = \frac{1}{2}\omega_0 t \rightarrow t = \frac{2\theta}{\omega_0} = \frac{2(5.1)}{15} s = 0.68\, s$

Angular Variables and Tangential Variables

Consider a point moving along a circular path in uniform circular motion ($\alpha = 0$). It starts at $\theta_0 = 0$ and makes an angle θ. The radius of the path is r and s is the arclength subtended by the angle θ. We know that,

$$s = \theta r \rightarrow s = (\omega t)r \rightarrow \frac{s}{t} = \omega r$$

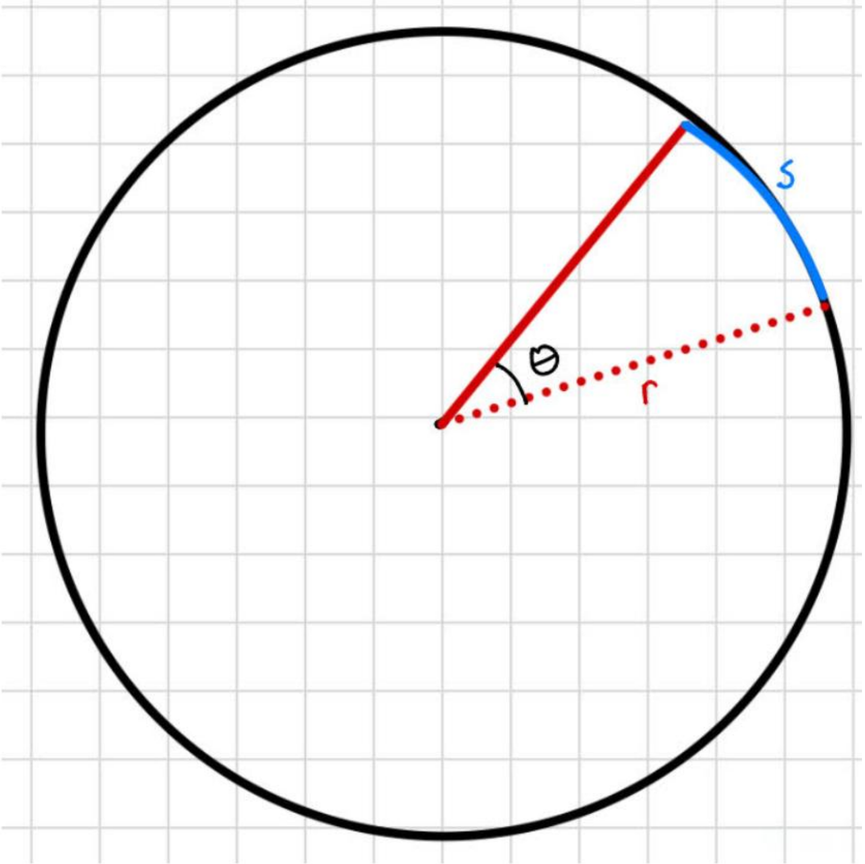
Because the motion is uniform, the speed is constant, so the average speed over any interval equals the instantaneous tangential speed: $v_T = s/t$.

$$\boxed{v_T = \omega r}$$

What about for constant but non-zero α? Consider the point starting from rest, our angular kinematics tell us,

$$\theta = \omega_o t + \frac{1}{2}\alpha t^2 = \frac{1}{2}\alpha t^2 \rightarrow r\theta = \frac{1}{2}r\alpha t^2 \rightarrow s = \frac{1}{2}r\alpha t^2$$

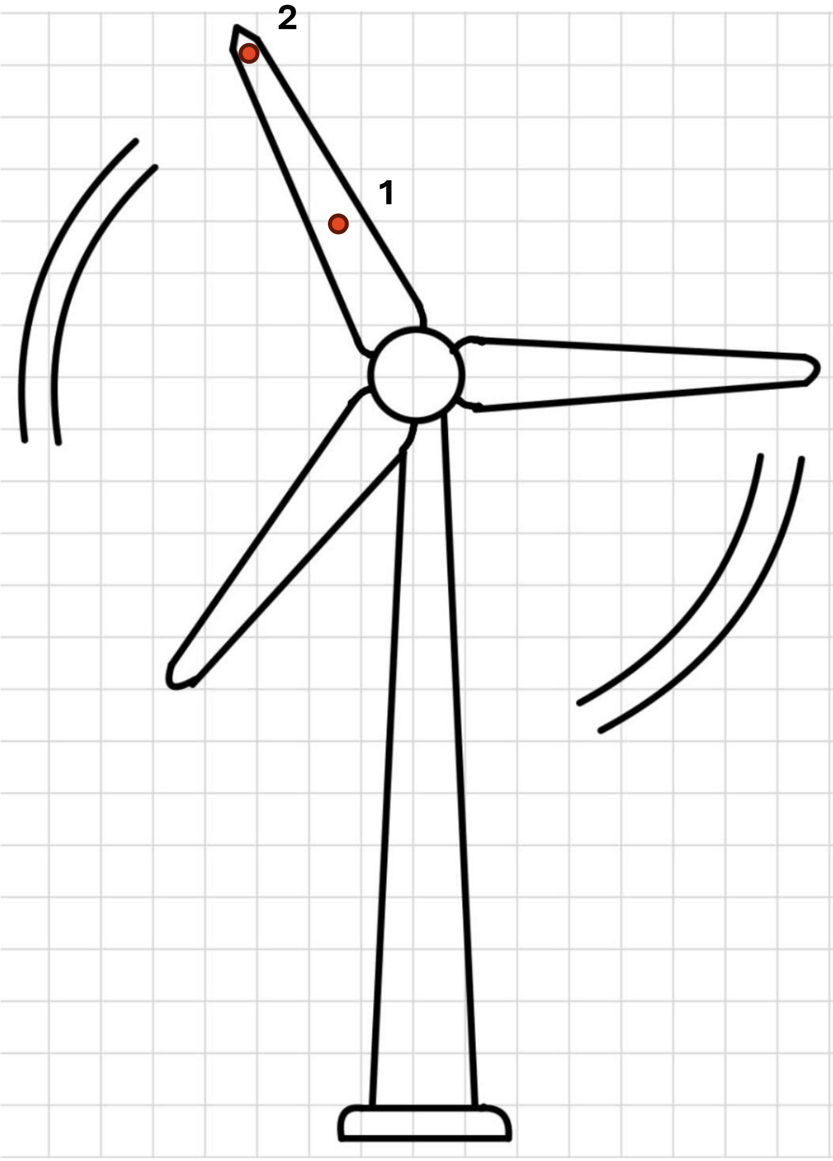
$$r\alpha = \frac{2s}{t^2} \rightarrow \boxed{a_T = r\alpha = \frac{2s}{t^2}}$$



Relates the tangential acceleration to the angular acceleration when the angular acceleration is constant.

Example 8.4

A wind turbine blade has an angular speed of 15.3 RPM and has an angular acceleration of 458 RPM^2. The distance from point 1 to the hub is 40m and from point 2 to the hub is 60m. For points 1 and 2 on the blade, find the magnitude of (a) the tangential speeds and (b) the tangential accelerations.



Solution: (a) Convert angular speed from RPM to rad/s:

$$\omega = \left(15.3\frac{rev}{min}\right)\left(\frac{1\ min}{60\ s}\right)\left(\frac{2\pi\ rad}{1\ rev}\right) = 1.60\ rad/s$$

The tangential speed at each point:

1. $v = r_1\omega = (40\ m)\left(1.60\frac{rad}{s}\right) = 64.0\ m/s$
2. $v = r_2\omega = (60\ m)\left(1.60\frac{rad}{s}\right) = 96.0\ m/s$

(b) Convert the angular speed to rev/s^2

$$\alpha = \left(458\frac{rev}{min^2}\right)\left(\frac{1\ min^2}{(60\ s)^2}\right)\left(\frac{2\pi}{1\ rev}\right) = 0.80\ rad/s^2$$

1. $a_T = r\alpha = (40\ m)(0.80\ rad/s^2) = 32\ m/s^2$
2. $a_T = r\alpha = (60\ m)(0.80\ rad/s^2) = 48\ m/s^2$

Centripetal Acceleration and Tangential Acceleration

The velocity is always tangent to the circle, so the radial component, v_r, is always zero: $\vec{v} = \vec{v}_T$

Since the particle is constrained to move along a circular path with a fixed radius, the radial component of velocity is zero so,

$$\vec{v} = \vec{v}_T + \cancel{\vec{v}_r} = \vec{v}_T$$

The instantaneous acceleration is,

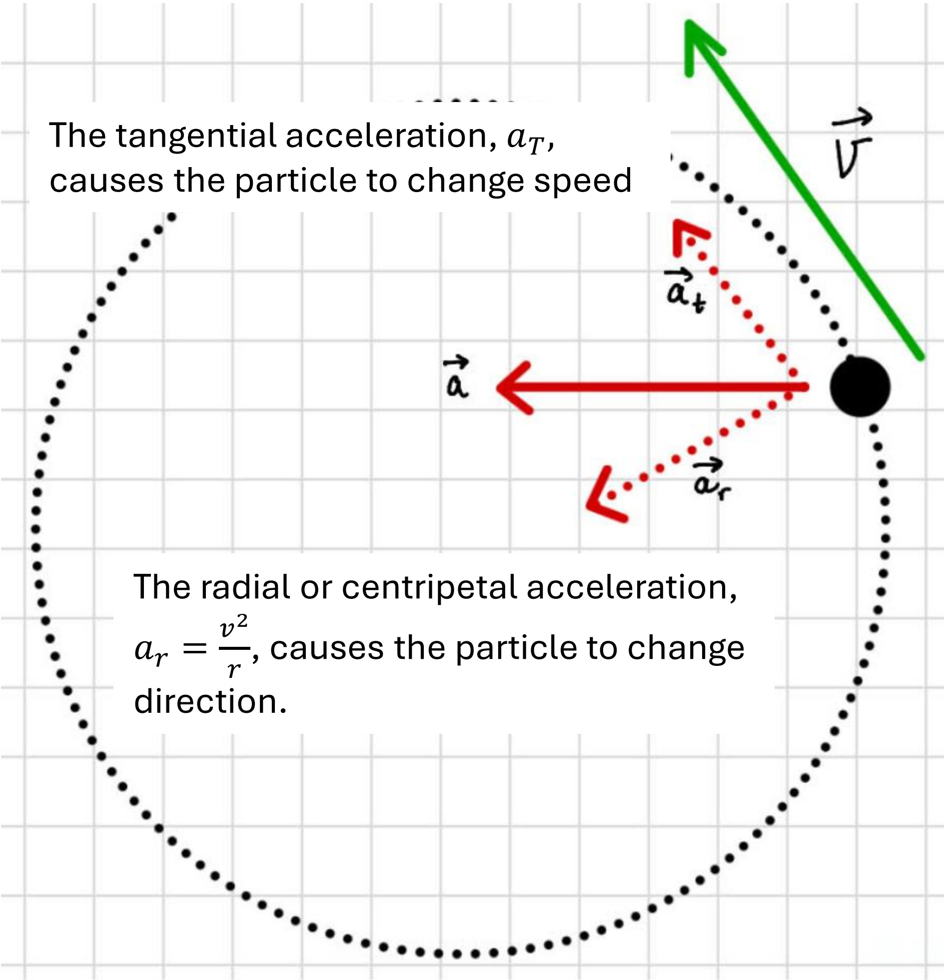
$$\vec{a} = \vec{a}_T + \vec{a}_r$$

With a magnitude of,

$$a = \sqrt{(r\alpha)^2 + (r\omega^2)^2} = r\sqrt{\alpha^2 + \omega^4}$$

Recalling

$$a_r = \frac{v^2}{r} = r\omega^2 \rightarrow \omega = \frac{v}{r}$$



Example 8.5: The Rogue Planet 1/2

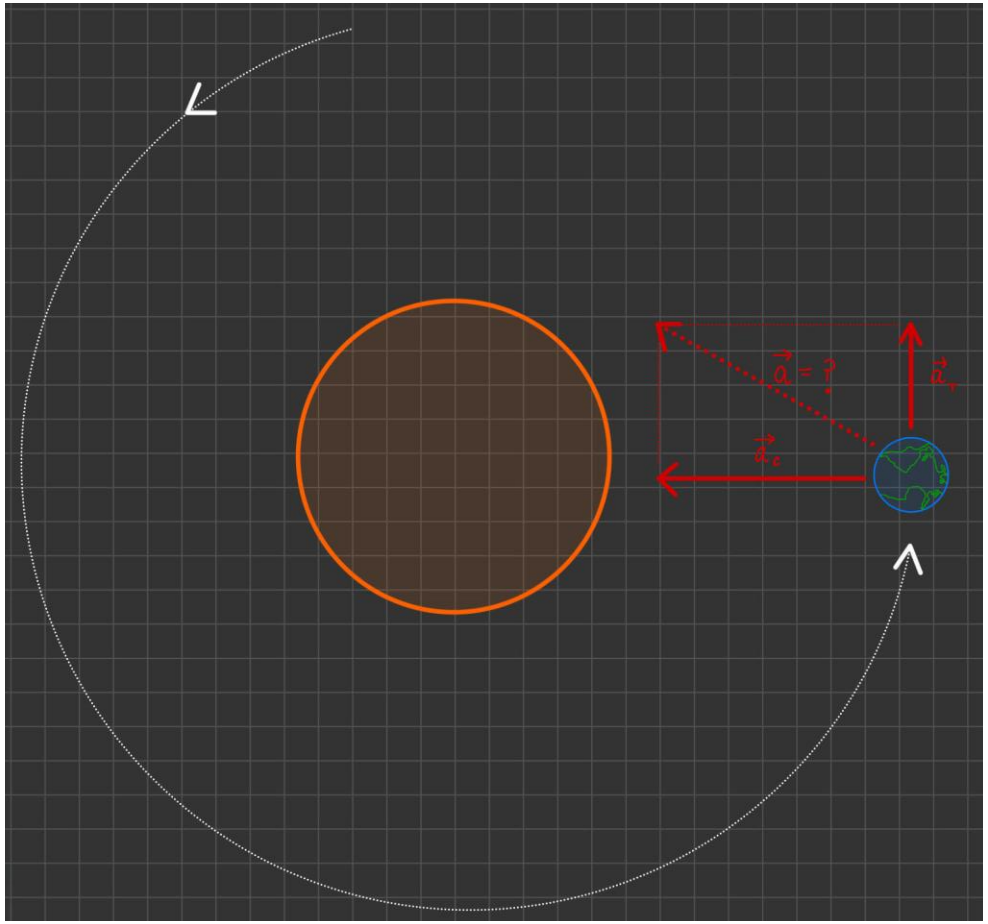
A planet orbits a star in a circular orbit of radius $r = 1.0 \times 10^{11}\ m$ around a star of mass $M = 1.5 \times 10^{30}\ kg$. Just before a close encounter ejects it from the system, a passing body induces a tangential acceleration of $a_t = 0.0040\ m/s^2$ forward along the orbit.

a. Compute the centripetal (radial) acceleration a_r just before release.

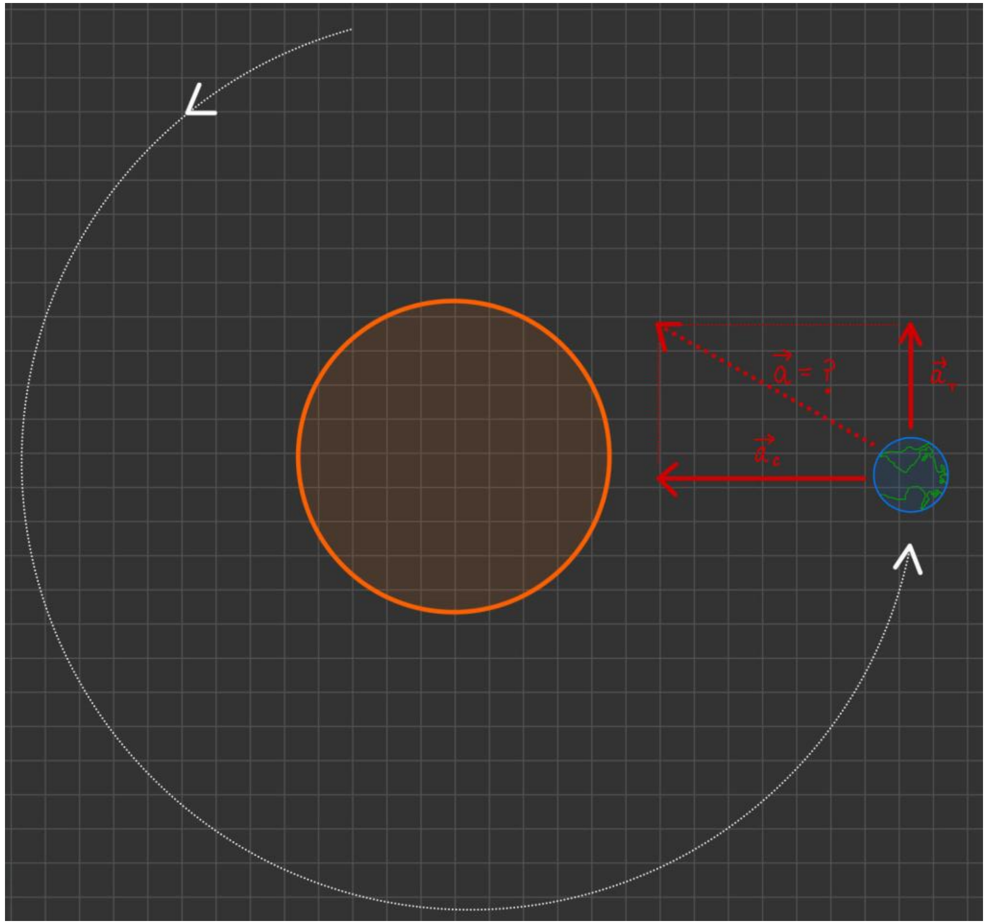
b. Find the magnitude of the total acceleration a.

c. Find the angle the total acceleration makes with the radial line to the star.

Use $G = 6.67 \times 10^{-11}\ Nm^2/kg^2$ .Assume the orbit is circular up to the instant of release.



Example 8.5: The Rogue Planet 2/2



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a. Compute the centripetal (radial) acceleration a_r just before release.

$$a_r = a_c = \frac{v^2}{r} = \frac{GM}{r^2} = \frac{(6.67 \times 10^{-11})(1.5 \times 10^{30})}{(1.0 \times 10^{11})} m/s^2$$
$$\approx 1.0 \times 10^{-2}\ m/s^2$$

b. Find the magnitude of the total acceleration a.

$$a = \sqrt{a_r^2 + a_T^2} = \sqrt{(1.0 \times 10^{-2})^2 + (4.0 \times 10^{-3})^2}\ m/s^2$$
$$\approx 1.08 \times 10^{-2} m/s^2$$

c. Find the angle the total acceleration makes with the radial line to the star.

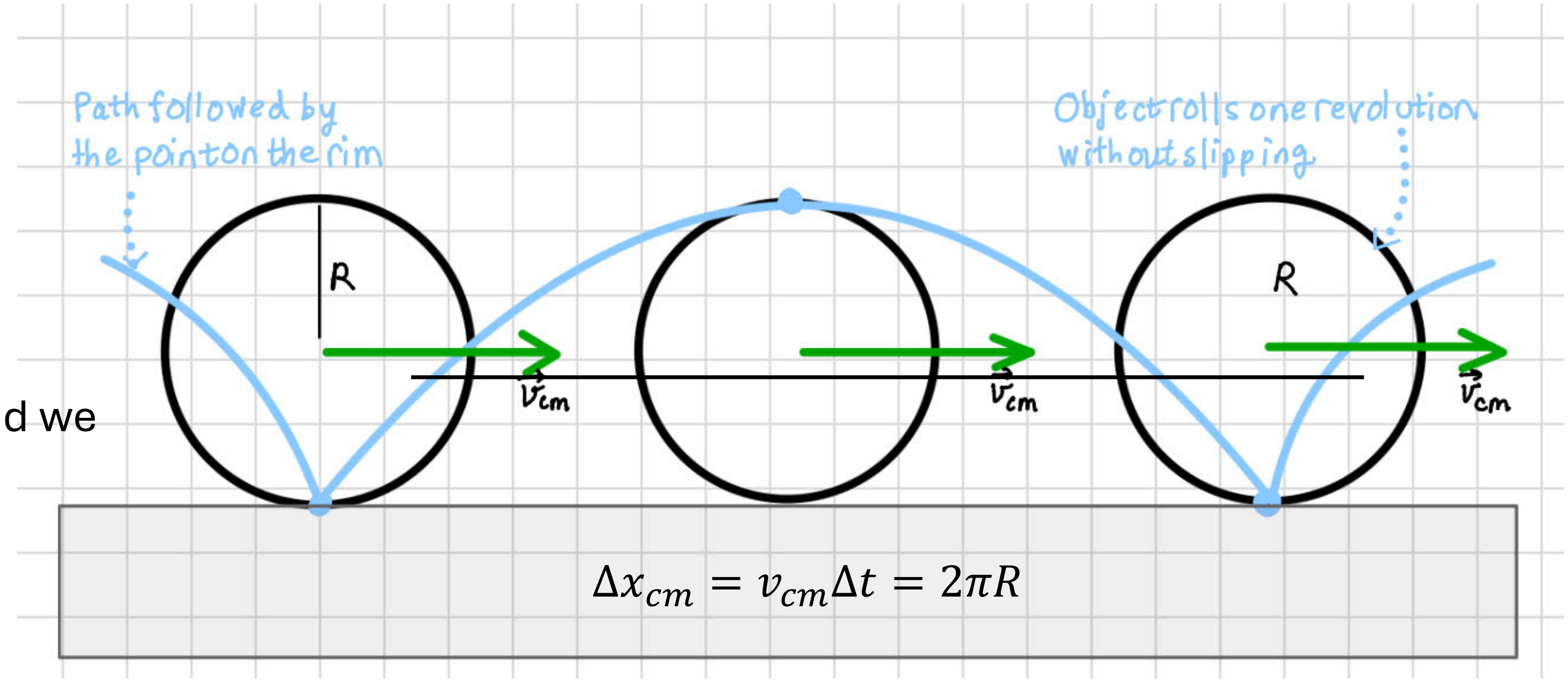
$$\theta = \tan^{-1}\frac{a_T}{a_r} = \tan^{-1}\left(\frac{0.004}{0.01}\right) \approx 21.8^0$$

Rolling Motion

Assume no slipping.

Rolling motion is both
- Translational motion
- Rotational motion

Note that $\Delta t = T$, is the period, and we know that $\omega = 2\pi/T$



$$\Delta x_{cm} = v_{cm}T = v_{cm}\frac{2\pi}{\omega} = 2\pi R \rightarrow v_{cm} = R\omega = v_T$$

Linear speed = Rotational Speed

The wheel also has a linear acceleration

$$a_{cm} = a_T = R\alpha$$

Concept Check 8.1: Jacked Up Truck

The speedometer of a truck is set to read the linear speed of the truck but uses a device that measures the angular speed of the rolling tires that came with the truck. However, the owner replaces the tires with larger-diameter versions. Does the reading on the speedometer after the replacement give a speed that is

a. Less than the true linear speed of the truck.

b. Equal to the true linear speed of the truck.

c. Greater than the true linear speed of the truck.